Proof of the Variational Principle for a Pair Hamiltonian Boson Model

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Abstract

We give a two parameter variational formula for the grand-canonical pressure of the *Pair Boson Hamiltonian* model. By using the Approximating Hamiltonian Method we provide a rigorous proof of this variational principle.

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1 Introduction

The first version of the *Pair Boson Hamiltonian* (PBH) model was proposed by Zubarev and Tserkovnikov in 1958 [1]. Their intention was to generalize the Bogoliubov model of the Weakly Imperfect Bose Gas [2] by including more terms from the total interaction, without losing the possibility of having an exact solution. We refer the reader to [3] and to [4] for a more recent discussion of this question.
The suggestion of Zubarev and Tserkovnikov [1] was to consider a truncated Hamiltonian which includes a *diagonal* term representing forward-scattering and exchange-scattering as well as a *non-diagonal* BCS-type interaction term. The model containing only the forward-scattering part of the interaction corresponds to the Mean-Field (or the Imperfect) Bose gas, see [4] and [5] for details. Using the same method as they had used earlier for the fermion BCS model [6], the authors give in [1] a "solution" of the PBH model. Later this Hamiltonian became the subject of very intensive analysis [7]-[9], leading essentially to the same conclusion as in [1], namely, that the PBH has the same thermodynamic properties as a certain *approximating* Hamiltonian quadratic in the creation and annihilation operators. Using this Hamiltonian which can be diagonalized by the canonical Bogoliubov transformation, its thermodynamic properties were investigated and it was shown to have some intriguing properties. One of these is possibility of the occurrence of two kinds of condensation, the standard one-particle Bose-Einstein condensation as well as a BCS-type pair condensation which may appear in two stages, see e.g. [10], [11]. Another one concerns the gap in the spectrum of "elementary excitations" [7]-[9]. In spite of fairly convincing arguments these papers did not prove rigorously that the above mentioned solution of the PBH model is exact. A mathematical treatment of the PBH model, related to representations of the Canonical Commutation Relations (*CCR*) appeared in [12].

In the present paper we give a variational formula for the pressure for the PBH model and provide a rigorous derivation of the formula. The latter yields the same expression for the pressure as was obtained in [1], the corresponding Euler-Lagrange equations coinciding with self-consistency equations studied in [1] and [7]-[12]. In an earlier paper [13] we conjectured that the pressure can be expressed as the *supremum* of a variational functional depending on two measures: a positive measure describing the particle density and a complex measure describing the pair density, similar to the Cooper pairs density in the BCS model. This confirmed the conclusion of [10], [11] about the coexistence of one-particle and pair condensates. The study in [13] was inspired by the *Large Deviation Principle* (LDP) developed for the analysis of boson systems in [14]-[17]. This method gives rigorous results for the pressure in the case of models with *diagonal* (commutative) boson interactions. A similar technique was developed in [18]-[23] based on the work [22], extending the LDP to *noncommutative* Mean-Field models (including the BCS one) with only bounded operators involved in Hamiltonians. Since neither of these methods apply to the PBH without extensive modifications, here we opted for the *Approximating Hamiltonian Method* (AHM) [24], which has been already successfully applied to many models, including some interacting boson models (see for example [4], [5], [25]).

There is renewed interest in the properties of the PBH interaction in the context of finite boson systems confined in a magneto-optic trap, see e.g. [26]-[28]. We do not discuss this aspect in the framework of our approach leaving it for future publications.

Now we turn to the exact formulation of the PBH model in its simplest form, that is, with *constant* pair and mean-field boson couplings [13].

Let $\Lambda \subset \mathbb{R}^\nu$ be a cube of volume $V = |\Lambda|$ centered at the origin. Then the kinetic energy operator for a particle of mass m confined to the cubic box Λ, that is the operator $-\Delta/2m$ with periodic boundary conditions, has eigenvalues $\epsilon(k) = \|k\|^2/2m$, $k \in \Lambda^* := \{2\pi s/V^{1/\nu} | s \in \mathbb{Z}^\nu\}$. Consider a system of identical bosons of mass m enclosed in Λ. For $k \in \Lambda^*$ let a_k^* and a_k be the usual boson creation and annihilation operators satisfying the *CCR* $[a_k\,, a_{k'}^*] = \delta_{k,k'}$ and let $N_k := a_k^* a_k$ be the k-mode particle number operator. The kinetic-energy operator T_Λ for the *Perfect Bose-gas*, can be expressed in the form $T_\Lambda := \sum_{k\in\Lambda^*} \epsilon(k) N_k$.
To introduce a *pairing term* in the Hamiltonian we shall need the operators

$$A_k = A_{-k} := a_k a_{-k}\,, \quad k \in \Lambda^*\,. \tag{1.1}$$

Let

$$N_\Lambda := \sum_{k\in\Lambda^*} N_k \quad \text{and} \quad \tilde{Q}_\Lambda := \sum_{k\in\Lambda^*} \tilde{\lambda}(k) A_k\,, \tag{1.2}$$

where the function $\tilde{\lambda} : \mathbb{R}^\nu \mapsto \mathbb{C}$ satisfies the following conditions:

$$|\tilde{\lambda}(k)| \le |\tilde{\lambda}(0)| = 1\,,\ \tilde{\lambda}(k) = \tilde{\lambda}(-k) \ \text{ for all } \ k \in \mathbb{R}^\nu\,,$$

there exists $\mathfrak{C} < \infty$ and $\delta > 0$ such that

$$|\tilde{\lambda}(k)| \le \frac{\mathfrak{C}}{1 + \|k\|^{\max(\nu,\,\nu/2+1)+\delta}} \tag{1.3}$$

for all $k \in \mathbb{R}^\nu$. Note that (1.3) implies that $\tilde{\lambda} \in L^1(\mathbb{R}^\nu)$ and that there exists $M < \infty$ such that

$$\mathfrak{m}_\Lambda := \sum_{k\in\Lambda^*} |\tilde{\lambda}(k)| \le MV, \tag{1.4}$$

$$\mathfrak{n}_\Lambda := \sum_{k\in\Lambda^*} \epsilon(k)|\tilde{\lambda}(k)|^2 \le MV, \tag{1.5}$$

and

$$\mathfrak{c}_\Lambda := \sup_{k\in\Lambda^*} \epsilon(k)|\tilde{\lambda}(k)|^2 \le M \tag{1.6}$$

for all $\Lambda \subset \mathbb{R}^\nu$.
Then for *constant* couplings u, v the PBH is defined by

$$H_\Lambda := T_\Lambda - \frac{u}{2V}\,\tilde{Q}_\Lambda^* \tilde{Q}_\Lambda + \frac{v}{2V}\,N_\Lambda^2. \tag{1.7}$$

Remark 1.1 *Let $\varphi := \arg\tilde{\lambda}(0)$ and $\lambda(k) := \tilde{\lambda}(k)e^{-\mathrm{i}\varphi}$. Then $\lambda(0) = 1$ and we can write H_Λ in the form*

$$H_\Lambda = T_\Lambda - \frac{u}{2V}\,Q_\Lambda^* Q_\Lambda + \frac{v}{2V}\,N_\Lambda^2 \tag{1.8}$$

with

$$Q_\Lambda := \sum_{k\in\Lambda^*} \lambda(k) A_k\,, \tag{1.9}$$

where $|\lambda(k)| \le \lambda(0) = 1$ for all $k \in \mathbb{R}^\nu$.

Remark 1.2 *We shall assume that $v > 0$ and $\alpha := v - u > 0$. The latter condition ensures the superstability of the model, see Theorem 2.1. Note that in the case $u \leq 0$ (BCS repulsion), the second condition $\alpha > 0$ is trivially satisfied. In [13] we have proved that the case $u \leq 0$ gives the same thermodynamics as the Mean-Field (MF) Bose-gas:*

$$H_{\Lambda}^{MF} := T_{\Lambda} + \frac{v}{2V} N_{\Lambda}^{2} . \tag{1.10}$$

Thus in deriving the variational formula we emphasize the case $u > 0$. We recall that this condition is necessary for nontrivial condensation of boson pairs, see e.g. [8]-[13]. We shall discuss the relation between these conditions and the thermodynamic properties of the model (1.8) in Section 5.

For the convenience of the reader we now state (without proof) the principal theorems and describe the the logical sequence used in proving the main result of this paper. We shall need the grand-canonical pressures for several *approximating* Hamiltonians. Recall that for an inverse temperature β and a chemical potential μ the the grand-canonical pressure for a system with Hamiltonian $\mathcal{H}_{\Lambda}$ is

$$\frac{1}{\beta V} \ln \operatorname{Tr} \exp \{-\beta(\mathcal{H}_{\Lambda} - \mu N_{\Lambda})\} \ . \tag{1.11}$$

For simplicity in the sequel we shall omit the thermodynamic variables β and μ and we shall write, for example, p_{Λ} for the grand-canonical pressure corresponding to the Hamiltonians H_{Λ}

$$p_{\Lambda} := \frac{1}{\beta V} \ln \operatorname{Tr} \exp \{-\beta(H_{\Lambda} - \mu N_{\Lambda})\} . \tag{1.12}$$

We shall denote the thermodynamic limit $\Lambda \uparrow \mathbb{R}^{\nu}$ by the symbol '$\lim_{\Lambda}$'.

Consider the *approximating* Hamiltonian

$$H_{\Lambda}^{(2)}(q,\rho) := T_{\Lambda} + v\rho N_{\Lambda} - \frac{1}{2}u(Q_{\Lambda}^{*}q + Q_{\Lambda}q^{*}) - \frac{V}{2}v\rho^{2} + \frac{V}{2}u|q|^{2} \ , \tag{1.13}$$

where $q \in \mathbb{C}$ and $\rho \in \mathbb{R}_{+}$ are variational parameters. The Hamiltonian $H_{\Lambda}^{(2)}(q,\rho)$ can be diagonalized and the corresponding pressure $p_{\Lambda}^{(2)}(q,\rho)$ can be calculated explicitly to give in the thermodynamic limit

$$\begin{aligned} p^{(2)}(q,\rho) : &= \lim_{\Lambda} p_{\Lambda}^{(2)}(q,\rho) \\ &= \int_{\mathbb{R}^{\nu}} \frac{d^{\nu}k}{(2\pi)^{\nu}} \left\{ -\frac{1}{\beta} \ln[1 - \exp(-\beta E(k,q,\rho))] - \frac{1}{2}\left(E(k,q,\rho) - f(k,\rho)\right) \right\} \\ &\quad -\frac{1}{2}u|q|^{2} + \frac{1}{2}v\rho^{2} \ , \end{aligned} \tag{1.14}$$

where

$$E(k,q,\rho) := \{f^{2}(k,\rho) - |h(k,q)|^{2}\}^{1/2} \ , \tag{1.15}$$

with

$$f(k,\rho) := \epsilon(k) - \mu + v\rho \quad \text{and} \quad h(k,q) := u\, q\, \lambda^{*}(k) \ . \tag{1.16}$$

Using (1.13) the Hamiltonian (1.8) can be written identically as

$$H_{\Lambda} = H_{\Lambda}^{(2)}(q,\rho) + H_{\Lambda}^{r}(q,\rho) \tag{1.17}$$

where

$$H^r_\Lambda(q,\rho) := -\frac{1}{2V}u(Q^*_\Lambda - Vq^*)(Q_\Lambda - Vq) + \frac{1}{2V}v(N_\Lambda - \rho)^2 \ . \tag{1.18}$$

The *main result* of this paper states that if the *variational parameters* q and ρ are chosen in an "optimal" way, then the contribution to the pressure arising from the *residual* term $H^r_\Lambda(q,\rho)$ vanishes in the thermodynamic limit.

Let us define the following function for $q \geq 0$ and $\rho \geq 0$

$$\sigma(q,\rho) := \inf_{k\in\mathbb{R}^\nu} \left(f(k,\rho) - |h(k,q)|\right) = v\rho - \mu - |u|q\,, \tag{1.19}$$

see (1.16).

Theorem 1.1 *The limiting pressure for the* PBH *model (1.8) with* $u > 0$ *(BCS attraction) has the form*

$$p := \lim_\Lambda p_\Lambda = \sup_{q\in\mathbb{C}} \inf_{\rho\geq 0} p^{(2)}(q,\rho) = \sup_{q\geq 0} \inf_{\rho\,:\,\sigma(q,\rho)\geq 0} p^{(2)}(q,\rho)\ , \tag{1.20}$$

while with $u \leq 0$ *(BCS repulsion) it has the form*

$$p := \lim_\Lambda p_\Lambda = \inf_{q\in\mathbb{C}} \inf_{\rho\geq 0} p^{(2)}(q,\rho) = \inf_{q\geq 0} \inf_{\rho\,:\,\sigma(q,\rho)\geq 0} p^{(2)}(q,\rho)\ . \tag{1.21}$$

Note that to obtain the approximating Hamiltonian (1.13), the term $-uQ^*_\Lambda Q_\Lambda/2V$ in (1.8) is replaced by $-u(Q^*_\Lambda q + Q_\Lambda q^*)/2 + Vu|q|^2/2$ and $vN^2_\Lambda/2V$ by $v\rho N_\Lambda - Vv\rho^2/2$.

We shall prove Theorem 1.1 in *two* steps. Here we describe these steps for $u > 0$ and before the end of the section we indicate the modifications necessary for the case $u \leq 0$.

The first step which we call the *first approximation* is to *linearize* the term $-uQ^*_\Lambda Q_\Lambda/2V$ in H_Λ. For technical reasons we need to add to our Hamiltonians some *source terms*. Therefore, we define for $\nu, \eta \in \mathbb{C}$

$$H_\Lambda(\nu,\eta) := H_\Lambda - (\nu Q^*_\Lambda + \nu^* Q_\Lambda) - \sqrt{V}\,(\eta a^*_0 + \eta^* a_0)\,, \tag{1.22}$$

and the *first approximating* Hamiltonian

$$\begin{aligned} H^{(1)}_\Lambda(q,\nu,\eta) := T_\Lambda + \frac{v}{2V}N^2_\Lambda - \frac{1}{2}u(Q^*_\Lambda q + Q_\Lambda q^*) + \frac{1}{2}Vu|q|^2 - \\ (\nu Q^*_\Lambda + \nu^* Q_\Lambda) - \sqrt{V}\,(\eta a^*_0 + \eta^* a_0)\,. \end{aligned} \tag{1.23}$$

From (1.22) and (1.23) we have

$$H_\Lambda(\nu,\eta) = H^{(1)}_\Lambda(q,\nu,\eta) + H^r_\Lambda(q)$$

where

$$H^r_\Lambda(q) = -\frac{1}{2V}u(Q^*_\Lambda - Vq^*)(Q_\Lambda - Vq) \leq 0. \tag{1.24}$$

First we show (see Section 3) that with the right choice of the parameter $q = \bar{q}$, the residual perturbation $H^r_\Lambda(\bar{q})$ does not contribute to $p_\Lambda(\nu,\eta)$, the pressure for the PBH (1.22) in the thermodynamic limit, i.e., the pressure corresponding to the Hamiltonian $H_\Lambda(\nu,\eta)$ coincides with the limit of $p^{(1)}_\Lambda(\bar{q},\nu,\eta)$, the pressure for $H^{(1)}_\Lambda(\bar{q},\nu,\eta)$:

Theorem 1.2 *For any ν and η with $|\nu| \leq 1$ and $|\eta| \leq 1$,*

$$\lim_{\Lambda} p_{\Lambda}(\nu, \eta) = \lim_{\Lambda} \sup_{q} p_{\Lambda}^{(1)}(q, \nu, \eta). \tag{1.25}$$

In particular

$$\lim_{\Lambda} p_{\Lambda}(\eta) = \lim_{\Lambda} \sup_{q} p_{\Lambda}^{(1)}(q, \eta). \tag{1.26}$$

where $p_{\Lambda}(\eta) := p_{\Lambda}(0, \eta)$ and $p_{\Lambda}^{(1)}(q, \eta) := p_{\Lambda}^{(1)}(q, 0, \eta)$ are the pressures corresponding to the Hamiltonians $H_{\Lambda}(\eta) := H_{\Lambda}(0, \eta)$ and $H_{\Lambda}^{(1)}(q, \eta) := H_{\Lambda}^{(1)}(q, 0, \eta)$ respectively.

Next, in Section 4 we study a *second approximating* Hamiltonian obtained from (1.23) by replacing the term $vN_{\Lambda}^2/2V$ by a linear term $v\rho N_{\Lambda} - Vv\rho^2/2$:

$$H_{\Lambda}^{(2)}(q, \rho, \eta) := T_{\Lambda} + v\rho N_{\Lambda} - \frac{1}{2}u(Q_{\Lambda}^* q + Q_{\Lambda} q^*) - \frac{V}{2}v\rho^2 + \frac{V}{2}u|q|^2 - \sqrt{V}\,(\eta a_0^* + \eta^* a_0)\,. \tag{1.27}$$

We denote the pressure corresponding to the Hamiltonian (1.27) by $\tilde{p}_{\Lambda}^{(2)}(q, \rho, \eta)$. Note that by (1.13) and (1.27) one has $H_{\Lambda}^{(2)}(q, \rho, 0) = H_{\Lambda}^{(2)}(q, \rho)$. We shall show in Lemma 4.1 that

$$\tilde{p}_{\Lambda}^{(2)}(q, \rho, \eta) = p_{\Lambda}^{(2)}(q, \rho) + |\eta|^2 \left\{ \frac{f(0,\rho) - |u||q| \cos(\theta - 2\psi)}{f^2(0,\rho) - u^2|q|^2} \right\}$$

where $\theta := \arg q$ and $\psi := \arg \eta$.

Our next theorem establishes a similar variational relation between the pressure $p_{\Lambda}(\eta)$ and $\tilde{p}_{\Lambda}^{(2)}(q, \rho, \eta)$:

Theorem 1.3

$$\lim_{\Lambda} p_{\Lambda}(\eta) = \lim_{\Lambda} \sup_{q\in\mathbb{C}} \inf_{\rho\geq 0} \tilde{p}_{\Lambda}^{(2)}(q, \rho, \eta) = \lim_{\Lambda} \sup_{q\geq 0} \inf_{\rho\geq 0} p_{\Lambda}^{(2)}(q, \rho, \eta)\,, \tag{1.28}$$

where for $q \geq 0$ we put

$$p_{\Lambda}^{(2)}(q, \rho, \eta) := \tilde{p}_{\Lambda}^{(2)}(q e^{\mathrm{i}(\pi+2\psi)}, \rho, \eta) = p_{\Lambda}^{(2)}(q, \rho) + \frac{|\eta|^2}{f(0, \rho) - uq}\,. \tag{1.29}$$

Note that the difference between the statement in Theorem 1.1 and that in Theorem 1.3 (apart from the η dependence) is that the thermodynamic limit is taken *after* taking the *infimum* over ρ and the *supremum* over q. In the next theorem we show that the order of the thermodynamic limit and taking the *infimum* and *supremum* can be reversed:

Theorem 1.4 *For $\eta \neq 0$,*

$$p(\eta) := \lim_{\Lambda} p_{\Lambda}(\eta) = \sup_{q\geq 0} \inf_{\rho\,:\,\sigma(q,\rho)\geq 0} p^{(2)}(q, \rho, \eta)\,, \tag{1.30}$$

where we put

$$p^{(2)}(q, \rho, \eta) := \lim_{\Lambda} p_{\Lambda}^{(2)}(q, \rho, \eta) = p^{(2)}(q, \rho) + \frac{|\eta|^2}{f(0, \rho) - uq}\,, \tag{1.31}$$

cf. expression (1.29).

In Lemma 4.5 we prove that $p = \lim_{\eta\to 0} p(\eta)$ so that Theorem 1.4 gives

$$p = \lim_{\eta\to 0} \sup_{q\geq 0} \inf_{\rho : \sigma(q,\rho)\geq 0} p^{(2)}(q,\rho,\eta). \tag{1.32}$$

Finally in Lemma 4.6 we prove that the order of the limit $\eta \to 0$ and taking the *infimum* and *supremum* can be reversed to yield the main result Theorem 1.1 for the BCS attraction.

The important difference for the *repulsive* case, $u < 0$, is that instead of (1.24) we now have

$$H^r_\Lambda(q) = -\frac{1}{2V} u(Q^*_\Lambda - Vq^*)(Q_\Lambda - Vq) \geq 0\ . \tag{1.33}$$

Therefore the first approximation (Section 3) should be constructed in the same way as the second approximation (Section 4). The proof of the second part of Theorem 1.1, (1.21), for $u \leq 0$ is given in Section 5 (f).

It is important to note that the variational formula conjectured in [13] has the same Euler-Lagrange equations as those given by Theorem 1.1. Thus the detailed study of these equations carried out in [13] applies to our result. In particular, this concerns the sequence of phase transitions in the PBH model (1.8) and the conditions for the *coexistence* of the *generalized* Bose condensation and the condensation of *boson pairs*, see also Section 5.

The paper is organized as follows. We start by proving in Section 2 that the PBH model (1.8) is superstable. In Sections 3 and 4 we shall assume that $u > 0$. Section 3 is devoted to establishing the first approximation giving the proof of Theorem 1.2. In Section 4 we turn to the second approximation giving the proof of Theorem 1.3 and the other results needed to obtain Theorem 1.1 for $u > 0$. Finally in Section 5 we discuss the variational problem as well as related open questions for all values of u and we finish the proof of Theorem 1.1 for $u \leq 0$. Some commutator relations are given in Appendix A and in Appendix B we give a bound needed in our proofs.

2 Superstability

In this section we establish the superstability of the PBH model (1.8). When $u \leq 0$ superstability is obvious. To prove it for $u > 0$ and $\alpha = v - u > 0$, we shall need the following lemma which is used in several other places in the paper.

Lemma 2.1 *The following inequality is satisfied*

$$Q^*_\Lambda Q_\Lambda \leq N^2_\Lambda + MVN_\Lambda. \tag{2.1}$$

Proof: The inequalities

$$\left(\lambda^*(k) a_{k'} a^*_k \pm \lambda^*(k') a^*_{-k'} a_{-k}\right)^* \left(\lambda^*(k) a_{k'} a^*_k \pm \lambda^*(k') a^*_{-k'} a_{-k}\right) \geq 0$$

and definition (1.1) imply that for $k \neq \{k', -k'\}$,

$$\begin{aligned} -(N_k + |\lambda(k)|)N_{k'} - (N_{-k'} + |\lambda(k')|)N_{-k} \\ \leq -|\lambda(k)|^2(N_k+1)N_{k'} - |\lambda(k')|^2(N_{-k'}+1)N_{-k} \\ \leq \lambda^*(k)\lambda(k') A^*_k A_{k'} + \lambda^*(k')\lambda(k) A^*_{k'} A_k \\ \leq |\lambda(k)|^2(N_k+1)N_{k'} + |\lambda(k')|^2(N_{-k'}+1)N_{-k} \\ \leq (N_k + |\lambda(k)|)N_{k'} + (N_{-k'} + |\lambda(k')|)N_{-k}\ . \end{aligned} \tag{2.2}$$

By (1.1) we also have

$$\begin{aligned} A_k^* A_k &= N_k N_{-k} \quad \text{for} \quad k \neq 0 \ , \\ A_0^* A_0 &= N_0(N_0 - 1) \leq N_0^2 \ . \end{aligned} \tag{2.3}$$

Then by (2.2) and (2.3) one gets

$$\begin{aligned} Q_\Lambda^* Q_\Lambda &= \sum_{\substack{k,k' \in \Lambda^*, \\ k \neq k', \ k \neq -k'}} \lambda^*(k)\lambda(k') A_k^* A_{k'} + 2\sum_{k \in \Lambda^*, \ k \neq 0} |\lambda(k)|^2 A_k^* A_k + |\lambda(0)|^2 A_0^* A_0 \\ &= \frac{1}{2} \sum_{\substack{k,k' \in \Lambda^*, \\ k \neq k', \ k \neq -k'}} (\lambda^*(k)\lambda(k') A_k^* A_{k'} + \lambda^*(k')\lambda(k) A_{k'}^* A_k) + 2\sum_{k \in \Lambda^*, \ k \neq 0} |\lambda(k)|^2 A_k^* A_k + |\lambda(0)|^2 A_0^* A_0 \\ &\leq \frac{1}{2} \sum_{\substack{k,k' \in \Lambda^*, \\ k \neq k', \ k \neq -k'}} ((N_k + |\lambda(k)|) N_{k'} + (N_{-k'} + |\lambda(k')|) N_{-k}) + 2\sum_{k \in \Lambda^*, \ k \neq 0} N_k N_{-k} + N_0^2 \\ &= \sum_{\substack{k,k' \in \Lambda^*, \\ k \neq k'}} N_k N_{k'} + \sum_{k \in \Lambda^*, \ k \neq 0} N_k N_{-k} + N_0^2 + \sum_{\substack{k,k' \in \Lambda^*, \\ k \neq k', \ k \neq -k'}} |\lambda(k)| N_{k'} \ . \end{aligned} \tag{2.4}$$

Using the inequality

$$N_k N_{-k} \leq \frac{1}{2} \left(N_k^2 + N_{-k}^2 \right) \ , \tag{2.5}$$

we get

$$\sum_{k \in \Lambda^*, \ k \neq 0} N_k N_{-k} \leq \sum_{k \in \Lambda^*, \ k \neq 0} N_k^2. \tag{2.6}$$

Thus (2.1) follows by (1.9) and (1.4). □

We now use the inequality (2.1) in Lemma 2.1 to prove superstability of the model (1.8).

Theorem 2.1 *The Hamiltonian (1.8) is superstable:*

$$H_\Lambda - \mu N_\Lambda \geq T_\Lambda + \frac{1}{2V} \alpha N_\Lambda^2 - (\mu + R) N_\Lambda \tag{2.7}$$

where $R := Mu/2$ *and* M *is defined by (1.4).*

Proof: From Lemma 2.1

$$\begin{aligned} H_\Lambda - \mu N_\Lambda &\geq T_\Lambda + \frac{1}{2V} (v - u) N_\Lambda^2 - (\mu + R) N_\Lambda \\ &= T_\Lambda + \frac{1}{2V} \alpha N_\Lambda^2 - (\mu + R) N_\Lambda. \end{aligned} \tag{2.8}$$

Since we are assuming that $\alpha > 0$, the estimate (2.8) implies superstability, see [29]. □

In the next two sections we develop the proofs for the variational formula for the pressure.

3 The First Approximation

Recall that the auxiliary Hamiltonians $H_\Lambda(\nu,\eta)$ and $H_\Lambda^{(1)}(q,\nu,\eta)$ are source dependent with $\nu,\eta\in\mathbb{C}$, see (1.22) and (1.23). Since later we shall let ν and η tend to zero, we can assume that $|\nu|\le 1$ and $|\eta|\le 1$. Because we are making the assumption on PBH (1.8) that $u>0$, it follows from (1.24) that $H_\Lambda^r(q)\le 0$.

Let $\nu\in\mathbb{C}$ and $\phi:=\arg(\nu^*\lambda(k))$. Then from

$$(a_k^*\pm e^{-\mathrm{i}\phi}a_{-k})(a_k\pm e^{\mathrm{i}\phi}a_{-k}^*)\ge 0$$

we get

$$-|\nu|(N_k+N_{-k}+|\lambda(k)|)\le \nu\lambda^*(k)A_k^*+\nu^*\lambda(k)A_k\le|\nu|(N_k+N_{-k}+|\lambda(k)|)\ . \tag{3.1}$$

Also

$$\sqrt{V}\,(\eta a_0^*+\eta^*a_0)=(a_0^*+\sqrt{V}\eta^*)(a_0+\sqrt{V}\eta)-a_0^*a_0-V|\eta|^2\ge -N_\Lambda-V|\eta|^2\ .$$

Therefore, by Theorem 2.1 one gets for $|\nu|\le 1$ and $|\eta|\le 1$, the estimate:

$$\begin{aligned} H_\Lambda(\nu,\eta)-\mu N_\Lambda &\ge H_\Lambda-\sum_{k\in\Lambda^*}(N_k+N_{-k}+|\lambda(k)|)-N_\Lambda-V-\mu N_\Lambda\\ &\ge H_\Lambda-(\mu+3)N_\Lambda-\mathfrak{m}_\Lambda-V\\ &\ge T_\Lambda+\frac{1}{4V}\alpha N_\Lambda^2-(\mu+3+R)N_\Lambda-(M+1)V\ . \end{aligned}\tag{3.2}$$

Since $H_\Lambda^r(q)\le 0$, we also have

$$\begin{aligned} H_\Lambda^{(1)}(q,\nu,\eta)-\mu N_\Lambda &\ge H_\Lambda(\nu,\eta)-\mu N_\Lambda\\ &\ge T_\Lambda+\frac{1}{4V}\alpha N_\Lambda^2-(\mu+3+R)N_\Lambda-(M+1)V\ . \end{aligned}\tag{3.3}$$

Proof of Theorem 1.2 :

For simplicity we shall prove this theorem for $\nu=0$. The proof for a general ν follows through verbatim by translation for $\nu\ne 0$. Clearly since $H_\Lambda^r\le 0$, it follows from (3.3) that for any q we have for the pressure of the PBH (1.22) the estimate from below:

$$p_\Lambda(\eta)\ge p_\Lambda^{(1)}(q,\nu=0,\eta)=p_\Lambda^{(1)}(q,\eta).$$

Also for any q one obviously has the estimate from above:

$$\begin{aligned} p_\Lambda(\eta) &= p_\Lambda^{(1)}(q,\eta)+\Big(p_\Lambda(\nu,\eta)-p_\Lambda^{(1)}(q,\nu,\eta))\Big)\\ &\qquad -(p_\Lambda(\nu,\eta)-p_\Lambda(\eta))+\Big(p_\Lambda^{(1)}(q,\nu,\eta))-p_\Lambda^{(1)}(q,\eta)\Big)\\ &\le \sup_{q'}p_\Lambda^{(1)}(q',\eta)+\Big(p_\Lambda(\nu,\eta)-p_\Lambda^{(1)}(q,\nu,\eta)\Big)\\ &\qquad -(p_\Lambda(\nu,\eta)-p_\Lambda(\eta))+\sup_{q'}\Big(p_\Lambda^{(1)}(\nu,q',\eta)-p_\Lambda^{(1)}(q',\eta)\Big)\,, \end{aligned}$$

and, therefore, we get

$$\begin{aligned}\sup_q p_\Lambda^{(1)}(q,\eta) \quad \leq \quad & p_\Lambda(\eta) \leq \sup_q p_\Lambda^{(1)}(q,\eta) + \inf_q \Big(p_\Lambda(\nu,\eta) - p_\Lambda^{(1)}(q,\nu,\eta)\Big) \\ & - (p_\Lambda(\nu,\eta) - p_\Lambda(\eta)) + \sup_q \Big(p_\Lambda^{(1)}(q,\nu,\eta) - p_\Lambda^{(1)}(q,\eta)\Big) . \end{aligned} \tag{3.4}$$

We shall prove in Lemma 3.1 that, if $\nu_\Lambda \to 0$ as $\Lambda \uparrow \mathbb{R}^\nu$, then

$$\liminf_\Lambda (p_\Lambda(\nu_\Lambda,\eta) - p_\Lambda(\eta)) = 0 \ , \tag{3.5}$$

and

$$\limsup_\Lambda \{\sup_q (p_\Lambda^{(1)}(q,\nu_\Lambda,\eta) - p_\Lambda^{(1)}(q,\eta))\} = 0 \ . \tag{3.6}$$

Next, with a *particular choice* of ν_Λ that tends to zero as $\Lambda \uparrow \mathbb{R}^\nu$, we shall show also that

$$\limsup_\Lambda \{\inf_q (p_\Lambda(\nu_\Lambda,\eta) - p_\Lambda^{(1)}(q,\nu_\Lambda,\eta))\} = 0 \ . \tag{3.7}$$

This last result (which is proved in Lemma 3.2) is much harder and requires the arguments developed in [24]. Putting these together we get

$$\lim_\Lambda p_\Lambda(\eta) = \lim_\Lambda \sup_q p_\Lambda^{(1)}(q,\eta) \, , \tag{3.8}$$

that proves Theorem 1.2 . □

We now prove the two lemmas quoted earlier.

Lemma 3.1

$$\liminf_\Lambda (p_\Lambda(\nu_\Lambda,\eta) - p_\Lambda(\eta)) = 0 \tag{3.9}$$

and

$$\limsup_\Lambda (p_\Lambda^{(1)}(q,\nu_\Lambda,\eta) - p_\Lambda^{(1)}(q,\eta)) = 0 \tag{3.10}$$

Proof: Writing $\nu = x + iy$, using the convexity of the pressure and (3.1) we get

$$\begin{aligned} p_\Lambda(\nu,\eta) - p_\Lambda(\eta) \quad \geq \quad & x\left(\frac{\partial}{\partial x} p_\Lambda(\nu,\eta)\right)\Bigg|_{\nu=0} + y\left(\frac{\partial}{\partial y} p_\Lambda(\nu,\eta)\right)\Bigg|_{\nu=0} \\ = \quad & \frac{1}{V}\left\langle \nu Q_\Lambda^* + \nu^* Q_\Lambda \right\rangle_{H_\Lambda(\eta)} \\ \geq \quad & -\frac{1}{V}|\nu| \sum_{k\in\Lambda^*} \left\langle N_k + N_{-k} + |\lambda(k)|\right\rangle_{H_\Lambda(\eta)} \\ \geq \quad & -\frac{1}{V}|\nu| \left(2\left\langle N_\Lambda\right\rangle_{H_\Lambda(\eta)} + \mathfrak{m}_\Lambda\right) \geq -K|\nu| \ , \end{aligned} \tag{3.11}$$

by (1.4) and Lemma B.1. Therefore if $\nu_\Lambda \to 0$ as $\Lambda \uparrow \mathbb{R}^\nu$, we get (3.9):

$$\liminf_\Lambda (p_\Lambda(\nu_\Lambda,\eta) - p_\Lambda(\eta)) = 0 \ . \tag{3.12}$$

Similarly one gets

$$\sup_q \left(p_\Lambda^{(1)}(q,\nu,\eta) - p_\Lambda^{(1)}(q,\eta)\right) \le \frac{1}{V}|\nu| \sup_q \left(2\,\langle N_\Lambda\rangle_{H^0_\Lambda(q,\nu,\eta)} + \mathfrak{m}_\Lambda\right) \le K|\nu|\ , \tag{3.13}$$

by (1.4), (3.3) and Lemma B.1. Thus

$$\limsup_\Lambda \{\sup_q (p_\Lambda^{(1)}(q,\nu_\Lambda,\eta) - p_\Lambda^{(1)}(q,\eta))\} = 0\ , \tag{3.14}$$

that implies (3.10). □

Lemma 3.2 *There exists a sequence* $\{\nu_\Lambda\}_\Lambda$ *that tends to* 0 *as* $\Lambda \uparrow \mathbb{R}^\nu$*, such that*

$$\limsup_\Lambda \inf_q \left(p_\Lambda(\nu_\Lambda,\eta) - p_\Lambda^{(1)}(q,\nu_\Lambda,\eta)\right) = 0\,. \tag{3.15}$$

Proof: Using the Bogoliubov convexity inequality [24]:

$$\frac{\mathrm{Tr}(A-B)e^B}{\mathrm{Tr}e^B} \le \ln \mathrm{Tr}e^A - \ln \mathrm{Tr}e^B \le \frac{\mathrm{Tr}(A-B)e^A}{\mathrm{Tr}e^A} \tag{3.16}$$

and (1.24) we get the estimate

$$0 \le p_\Lambda(\nu,\eta) - p_\Lambda^{(1)}(q,\nu,\eta) \le \frac{1}{2V^2} u\, \langle (Q_\Lambda^* - Vq^*)(Q_\Lambda - Vq)\rangle_{H_\Lambda(\nu,\eta)}\,.$$

Let $\delta Q_\Lambda(\nu,\eta) := Q_\Lambda - \langle Q_\Lambda\rangle_{H_\Lambda(\nu,\eta)}$ and let

$$\Delta_\Lambda(\nu,\eta) := \langle \delta Q_\Lambda^*(\nu,\eta)\ \delta Q_\Lambda(\nu,\eta)\rangle_{H_\Lambda(\nu,\eta)} \ge 0.$$

Then

$$\inf_q \left(p_\Lambda(\nu,\eta) - p_\Lambda^{(1)}(q,\nu,\eta)\right) \le \frac{u}{2V^2}\Delta_\Lambda(\nu,\eta)\ . \tag{3.17}$$

We want to obtain an estimate for $\Delta_\Lambda(\nu,\eta)$ in terms of ν and V.

Let

$$D_\Lambda(\nu,\eta) := (\delta Q_\Lambda^*(\nu,\eta),\ \delta Q_\Lambda(\nu,\eta))_{H_\Lambda(\nu,\eta)}\,, \tag{3.18}$$

where $(\cdot\ ,\ \cdot)_H$ denotes the Bogoliubov-Duhamel *inner product* with respect to the Hamiltonian H, see for example [24] or [25]. Using the Ginibre inequality (e.g. (2.10) in [25]) we get

$$\begin{aligned}\Delta_\Lambda(\nu,\eta) &\le \frac{1}{2}\langle \delta Q_\Lambda^*(\nu,\eta)\ \delta Q_\Lambda(\nu,\eta) + \delta Q_\Lambda^*(\nu,\eta)\ \delta Q_\Lambda(\nu,\eta)\rangle_{H_\Lambda(\nu,\eta)} \\ &\le D_\Lambda(\nu,\eta) + \frac{1}{2}\{\beta D_\Lambda(\nu,\eta)\}^{1/2}\left\{\langle [Q_\Lambda^*,\ [H_\Lambda(\nu,\eta) - \mu N_\Lambda, Q_\Lambda]]\rangle_{H_\Lambda(\nu,\eta)}\right\}^{1/2}.\end{aligned}$$

We shall show in Appendix A that there is a real number C such that

$$\langle [Q_\Lambda^*,\ [H_\Lambda(\nu,\eta) - \mu N_\Lambda, Q_\Lambda]]\rangle_{H_\Lambda(\nu,\eta)} \le C\,V^{3/2}\ .$$

Thus

$$\Delta_\Lambda(\nu,\eta) \le D_\Lambda(\nu,\eta) + (C\beta)^{1/2}\left\{V^{3/2} D_\Lambda(\nu,\eta)\right\}^{1/2}. \tag{3.19}$$

From the definition of the Bogoliubov-Duhamel inner product we have

$$D_{\Lambda}(\nu,\eta) = V\frac{\partial^2}{\partial\nu\partial\nu^*}p_{\Lambda}(\nu,\eta) = \frac{V}{4}\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right\}p_{\Lambda}(\nu,\eta)\ .$$

Here we consider the pressure $p_{\Lambda}(\nu,\eta)$ as a function of two real variables, $x = \mathfrak{Re}\,\nu$ and $y = \mathfrak{Im}\,\nu$. Since $u > 0$, then following the *Approximating Hamiltonian Method* for *attractive* interactions [24] we consider the integral

$$I_{\Lambda}(\delta) := \int_{[-\delta,\delta]^2} dx\, dy \frac{\partial^2}{\partial x^2}p_{\Lambda}(\nu,\eta)\,.$$

With $\nu_+ := \delta + iy$ and $\nu_- := -\delta + iy$, this integral is equal to

$$\begin{aligned} I_{\Lambda}(\delta) &= \int_{[-\delta,\delta]} dy \left\{\frac{\partial}{\partial x}p_{\Lambda}(\nu_+,\eta) - \frac{\partial}{\partial x}p_{\Lambda}(\nu_-,\eta)\right\} \\ &= \frac{1}{V}\int_{[-\delta,\delta]} dy \left\{\langle Q_{\Lambda}+Q_{\Lambda}^*\rangle_{H_{\Lambda}(\nu_+,\eta)} - \langle Q_{\Lambda}+Q_{\Lambda}^*\rangle_{H_{\Lambda}(\nu_-,\eta)}\right\}. \end{aligned}$$

Then by (3.1) one gets

$$|I_{\Lambda}(\delta)| \le \frac{2}{V}\int_{[-\delta,\delta]} dy \left\{\left\langle \tilde{N}_{\Lambda}\right\rangle_{H_{\Lambda}(\nu_+,\eta)} + \left\langle \tilde{N}_{\Lambda}\right\rangle_{H_{\Lambda}(\nu_-,\eta)}\right\},$$

where $\tilde{N}_{\Lambda} := \sum_{k\in\Lambda^*}(N_k + N_{-k} + |\lambda(k)|)/2$. Since by (3.2) and Lemma B.1, the expectation $\langle N_{\Lambda}/V\rangle_{H_{\Lambda}(\nu,\eta)}$ is bounded uniformly in ν and in V, we obtain the estimate

$$\begin{aligned} &\left|\int_{[-\delta,\delta]^2} dx\, dy \frac{\partial^2}{\partial x^2}p_{\Lambda}(\nu,\eta)\right| \le \\ &\frac{2}{V}\int_{[-\delta,\delta]} dy \left\{\langle N_{\Lambda}\rangle_{H_{\Lambda}(\nu_+,\eta)} + \langle N_{\Lambda}\rangle_{H_{\Lambda}(\nu_-,\eta)} + \mathfrak{m}_{\Lambda}\right\} \le 2\tilde{C}\delta\,. \end{aligned}$$

Similarly one gets the estimate

$$\left|\int_{[-\delta,\delta]^2} dx\, dy \frac{\partial^2}{\partial y^2}p_{\Lambda}(\nu,\eta)\right| \le 2\tilde{C}\delta\,.$$

These give

$$\int_{[-\delta,\delta]^2} dx\, dy D_{\Lambda}(\nu,\eta) \le \tilde{C}V\delta\,. \tag{3.20}$$

Since the integrand is continuous, by the *integral mean-value* theorem there exists a sequence $\{\nu_{\Lambda}\}_{\Lambda}$ with $|\nu_{\Lambda}| \le \delta$ such that

$$\int_{[-\delta,\delta]^2} dx\, dy D_{\Lambda}(\nu,\eta) = (2\delta)^2 D_{\Lambda}(\nu_{\Lambda},\eta)\,.$$

The last equation and inequality (3.20) imply that

$$D_{\Lambda}(\nu_{\Lambda},\eta) \le \frac{\tilde{C}V}{4\delta}\,,$$

which together with (3.19) give the estimate

$$\frac{1}{V^2}\,\Delta_\Lambda(\nu_\Lambda,\eta) \le \frac{\tilde{C}}{4V\delta} + \frac{(\tilde{C}C\beta)^{1/2}}{2V^{3/4}\delta^{1/2}}\,.$$

Choosing $\delta = \delta_\Lambda$ such that $\delta_\Lambda \to 0$, but $V\,\delta_\Lambda \to \infty$, we get

$$\lim_\Lambda \frac{1}{V^2}\,\Delta_\Lambda(\nu_\Lambda,\eta) = 0\,.$$

By (3.17) this completes the proof of the lemma. □

This proves the first approximation. In the next section we deal with the second one.

4 The Second Approximation

Note that from definitions (1.23) and (1.27) of the *first* and the *second* approximating Hamiltonians, $H_\Lambda^{(1)}(q,\nu,\eta)$ and $H_\Lambda^{(2)}(q,\rho,\eta)$, respectively, it follows that

$$H_\Lambda^{(1)}(q,\nu=0,\eta) - H_\Lambda^{(2)}(q,\rho,\eta) = \frac{1}{2V}\,v(N_\Lambda-\rho)^2 \ge 0\,. \tag{4.1}$$

Later in this section we shall show (see Lemma 4.1 and Remark 4.1) that

$$\tilde{p}_\Lambda^{(2)}(q,\rho,\eta) \le \tilde{p}_\Lambda^{(2)}(|q|e^{\mathrm{i}(\pi+2\psi)},\rho,\eta) = p_\Lambda^{(2)}(|q|,\rho,\eta)\,. \tag{4.2}$$

In Lemma 4.2 we prove that for each $q \ge 0$ there is a unique density $\rho = \bar{\rho}_\Lambda(q,\eta) > 0$, such that

$$p_\Lambda^{(2)}(q,\bar{\rho}_\Lambda(q,\eta),\eta) = \inf_\rho p_\Lambda^{(2)}(q,\rho,\eta)\,. \tag{4.3}$$

We can also show (Lemma 4.3) that there is at least one $q = \bar{q}_\Lambda(\eta) > 0$, such that

$$p_\Lambda^{(2)}(\bar{q}_\Lambda,\bar{\rho}_\Lambda(\bar{q}_\Lambda),\eta) = \sup_q p_\Lambda^{(2)}(q,\bar{\rho}_\Lambda(q),\eta) = \sup_q \inf_\rho p_\Lambda^{(2)}(q,\rho,\eta)\,. \tag{4.4}$$

For the sake of simplicity below we shall omit the variable η, and we put

$$\bar{\rho}_\Lambda(q,\eta) := \bar{\rho}_\Lambda(q) \quad \text{and} \quad \bar{q}_\Lambda(\eta) := \bar{q}_\Lambda\,.$$

Finally, we shall show in Lemma 4.4 that if $\eta \ne 0$, then

$$\lim_\Lambda \{p_\Lambda^{(2)}(\bar{q}_\Lambda,\bar{\rho}_\Lambda(\bar{q}_\Lambda),\eta) - p_\Lambda^{(1)}(\bar{q}_\Lambda e^{\mathrm{i}(\pi+2\psi)},\eta)\} = 0\,. \tag{4.5}$$

We start by proving Theorem 1.3, assuming the results of Lemmas 4.1 - 4.4, which we prove later.

Proof of Theorem 1.3 :

We have to prove the limit (1.28) i.e. that

$$p(\eta) := \lim_\Lambda p_\Lambda(\eta) = \lim_\Lambda p_\Lambda^{(2)}(\bar{q}_\Lambda,\bar{\rho}_\Lambda(\bar{q}_\Lambda),\eta)\,. \tag{4.6}$$

First, by (4.1) and (4.2) we have for all values of the variational parameters q, ρ and the source parameter η that

$$p_\Lambda^{(1)}(q,\eta) := p_\Lambda^{(1)}(q,\nu=0,\eta) \le \tilde{p}_\Lambda^{(2)}(q,\rho,\eta) \le p_\Lambda^{(2)}(|q|,\rho,\eta)\,.$$

Therefore,

$$p_\Lambda^{(1)}(q,\eta) \le \inf_\rho p_\Lambda^{(2)}(|q|,\rho,\eta) = p_\Lambda^{(2)}(|q|,\bar{\rho}_\Lambda(|q|),\eta)$$

and thus by definition (1.29) we obtain

$$\sup_q p_\Lambda^{(1)}(q,\eta) \le \sup_q p_\Lambda^{(2)}(|q|,\bar{\rho}_\Lambda(|q|),\eta) = \sup_{q\ge 0} p_\Lambda^{(2)}(q,\bar{\rho}_\Lambda(q),\eta) = p_\Lambda^{(2)}(\bar{q}_\Lambda,\bar{\rho}_\Lambda(\bar{q}_\Lambda),\eta)\,.$$

This estimate implies that

$$\lim_\Lambda \sup_q p_\Lambda^{(1)}(q,\eta) \le \lim_\Lambda p_\Lambda^{(2)}(\bar{q}_\Lambda,\bar{\rho}_\Lambda(\bar{q}_\Lambda),\eta)\,. \tag{4.7}$$

On the other hand for all η we obviously have

$$\begin{aligned}\sup_q p_\Lambda^{(1)}(q,\eta) \ge p_\Lambda^{(1)}(\bar{q}_\Lambda e^{\mathrm{i}(\pi+2\psi)},\eta) &= p_\Lambda^{(2)}(\bar{q}_\Lambda,\bar{\rho}_\Lambda(\bar{q}_\Lambda),\eta) \\ &- \Big(p_\Lambda^{(2)}(\bar{q}_\Lambda,\bar{\rho}_\Lambda(\bar{q}_\Lambda),\eta) - p_\Lambda^{(1)}(\bar{q}_\Lambda e^{\mathrm{i}(\pi+2\psi)},\eta)\Big)\,.\end{aligned} \tag{4.8}$$

Now the limit (4.5) and the estimate (4.8) imply that

$$\lim_\Lambda \sup_q p_\Lambda^{(1)}(q,\eta) \ge \lim_\Lambda p_\Lambda^{(2)}(\bar{q}_\Lambda,\bar{\rho}_\Lambda(\bar{q}_\Lambda),\eta). \tag{4.9}$$

Taking into account (4.7) and (4.9) we get

$$\lim_\Lambda \sup_q p_\Lambda^{(1)}(q,\eta) = \lim_\Lambda p_\Lambda^{(2)}(\bar{q}_\Lambda,\bar{\rho}_\Lambda(\bar{q}_\Lambda),\eta)\,.$$

Combining this result with Theorem 1.2 we get (4.6), i.e. the proof of Theorem 1.3. □

Now we return to proof of the lemmas quoted earlier.

Lemma 4.1 *Let the functions f and h and the spectral function $E(k,q,\rho)$ be as defined in (1.16) and (1.15) respectively.*
(i) If $f(0,\rho) > u|q| \ge 0$, the pressure $\tilde{p}_\Lambda^{(2)}(q,\rho,\eta)$ corresponding to $H_\Lambda^{(2)}(q,\rho,\eta)$ is given by

$$\begin{aligned}\tilde{p}_\Lambda^{(2)}(q,\rho,\eta) &= -\frac{1}{\beta V}\sum_{k\in\Lambda^*}\ln\{1-\exp(-\beta E(k,q,\rho))\} - \frac{1}{2V}\sum_{k\in\Lambda^*}(E(k,q,\rho)-f(k,\rho)) \\ &+ |\eta|^2\left\{\frac{f(0,\rho)-|uq|\cos(\theta-2\psi)}{f^2(0,\rho)-u^2|q|^2}\right\} - \frac{1}{2}u|q|^2 + \frac{1}{2}v\rho^2\,,\end{aligned} \tag{4.10}$$

where $\theta = \arg q$ and $\psi = \arg\eta$.

(ii) If $f(0,\rho) \le u|q|$, then $\tilde{p}_\Lambda^{(2)}(q,\rho,\eta)$ is infinite.

Proof: (i) By (1.16) and (1.27) we can write $H^{(2)}_\Lambda(q,\rho,\eta)-\mu N_\Lambda$ in the form

$$\begin{aligned} H^{(2)}_\Lambda(q,\rho,\eta)-\mu N_\Lambda &= \sum_{k\in\Lambda^*}\{f(k,\rho)a^*_k a_k - \frac{1}{2}\left(h(k,q)a^*_k a^*_{-k} + h^*(k,q)a_{-k}a_k\right)\} \\ &- \sqrt{V}\,(\eta a^*_0 + \eta^* a_0) + VW(q,\rho)\,, \end{aligned}$$

where

$$W(q,\rho) = \frac{1}{2}u|q|^2 - \frac{1}{2}v\rho^2\,.$$

Let $q\lambda^*(k) = |q\lambda^*(k)|e^{\mathrm{i}\theta(k)}$. Then with $a_k = \tilde{a}_k e^{\mathrm{i}\theta(k)/2}$, for $k\in\Lambda^*$, one gets

$$\begin{aligned} H^{(2)}_\Lambda(q,\rho,\eta)-\mu N_\Lambda &= \sum_{k\in\Lambda^*}\{f(k,\rho)\tilde{a}^*_k\tilde{a}_k - \frac{1}{2}|h(k,q)|\left(\tilde{a}^*_k\tilde{a}^*_{-k} + \tilde{a}_{-k}\tilde{a}_k\right)\} \\ &- \sqrt{V}\left(\eta e^{-\mathrm{i}\theta/2}\tilde{a}^*_0 + \eta^* e^{\mathrm{i}\theta/2}\tilde{a}_0\right) + VW(q,\rho)\,, \end{aligned} \tag{4.11}$$

where $\theta = \arg q = \theta(0)$.
Note that if $f(0,\rho) > u|q| \geq 0$, then $f(k,\rho) > |h(k,q)| \geq 0$ for all $k\in\Lambda^*$, so that $E(k,q,\rho)$ is well-defined and *positive*, see (1.15). Let

$$x_k^2 = \frac{1}{2}\left\{\frac{f(k,\rho)}{E(k,q,\rho)} + 1\right\} \quad \text{and} \quad y_k^2 = \frac{1}{2}\left\{\frac{f(k,\rho)}{E(k,q,\rho)} - 1\right\}. \tag{4.12}$$

Then the *canonical Bogoliubov* transformation: $\tilde{a}_k = x_k\alpha_k - y_k\alpha^*_{-k}$, gives

$$\begin{aligned} &H^{(2)}_\Lambda(q,\rho,\eta)-\mu N_\Lambda = \sum_{k\in\Lambda^*} E(k,q,\rho)\alpha^*_k\alpha_k - \sqrt{V}\,(\xi\alpha^*_0 + \xi^*\alpha_0) \\ &+\frac{1}{2}\sum_{k\in\Lambda^*}(E(k,q,\rho) - f(k,\rho)) + VW(q,\rho)\,, \end{aligned} \tag{4.13}$$

where α^*_k and α_k , $k\in\Lambda^*$, are boson creation and annihilation operators and

$$\xi = \eta\ x_0 e^{-\mathrm{i}\theta/2} - \eta^* y_0 e^{\mathrm{i}\theta/2}\ .$$

We note that

$$|\xi|^2 = |\eta|^2\,\frac{f(0,\rho) - |uq|\cos(\theta - 2\psi)}{E(0,q,\rho)}\quad .$$

From the diagonal form of $H^{(2)}_\Lambda(q,\rho,\eta)-\mu N_\Lambda$ in (4.13) we get the pressure (4.10).
(ii) Now let $f(0,\rho) < u|q|$. Then the quadratic Hamiltonian (4.11) is not bounded from below. This means that the trace in (1.12) is divergent and therefore the pressure $\tilde{p}^{(2)}_\Lambda(q,\rho,\eta)$ is infinite. If $f(0,\rho) = u|q|$, then by definitions (1.16) and the conditions on $\tilde{\lambda}(k)$ at least the zero-mode term of the Hamiltonian (4.11) is *not* positive. This again implies that the trace in expression (1.12) diverges. □

Remark 4.1 *From the explicit formula (4.10) it follows that*

$$\tilde{p}^{(2)}_\Lambda(q,\rho,\eta) \leq \tilde{p}^{(2)}_\Lambda(|q|e^{\mathrm{i}(\pi+2\psi)},\rho,\eta) = p^{(2)}_\Lambda(|q|,\rho,\eta)\,.$$

Recall that by (1.29) and (4.10) one gets for $q\geq 0$

$$\begin{aligned} p^{(2)}_\Lambda(q,\rho,\eta) &= -\frac{1}{\beta V}\sum_{k\in\Lambda^*}\ln\{1-\exp(-\beta E(k,q,\rho))\} - \frac{1}{2V}\sum_{k\in\Lambda^*}(E(k,q,\rho) - f(k,\rho)) \\ &+ \frac{|\eta|^2}{f(0,\rho) - uq} - \frac{1}{2}uq^2 + \frac{1}{2}v\rho^2\,. \end{aligned} \tag{4.14}$$

Lemma 4.2 *Let $\eta \neq 0$. Then there are numbers $0 < \tilde{\rho}_1(q,\eta) < \tilde{\rho}_2(q,\eta) < \infty$, such that the infimum of $p_\Lambda^{(2)}(q,\rho,\eta)$ over ρ is attained in the interval $(\tilde{\rho}_1(q,\eta), \tilde{\rho}_2(q,\eta))$ and if $\bar{\rho}_\Lambda(q)$ is a value of ρ at which the infimum is attained, then $\partial p_\Lambda^{(2)}(q,\bar{\rho}_\Lambda(q),\eta)/\partial\rho = 0$. Moreover, if $0 < q_0 < \infty$, then*

$$\inf_{q\leq q_0} (v\tilde{\rho}_1(q,\eta) - (\mu+uq)_+) > 0 \qquad \text{and} \qquad \sup_{q\leq q_0} \tilde{\rho}_2(q,\eta) < \infty \ ,$$

where $s_\pm := \max(0,\pm s)$ for $s \in \mathbb{R}$.

Proof: By (4.14) we have

$$\frac{\partial p_\Lambda^{(2)}}{\partial\rho}(q,\rho,\eta) = -\frac{v}{V}\sum_{k\in\Lambda^*}\left\{\frac{1}{\exp(\beta E(k,q,\rho))-1}\,\frac{f(k,\rho)}{E(k,q,\rho)} + \frac{1}{2}\left(\frac{f(k,\rho)}{E(k,q,\rho)}-1\right)\right\}$$
$$-\frac{v|\eta|^2}{(f(0,\rho)-uq)^2} + v\rho \ . \tag{4.15}$$

From (4.15) we get

$$\frac{\partial p_\Lambda^{(2)}}{\partial\rho}(q,\rho,\eta) \leq -\frac{v|\eta|^2}{(f(0,\rho)-uq)^2} + v\rho \ .$$

Let $x := v\rho - (\mu+uq)_+$. Using the identity $\mu + uq = (\mu+uq)_+ - (\mu+uq)_-$ we obtain

$$\frac{\partial p_\Lambda^{(2)}}{\partial\rho}(q,\rho,\eta) \leq -\frac{v|\eta|^2}{((\mu+uq)_- + x)^2} + (\mu+uq)_+ + x \ .$$

As $x \to 0$, the right-hand side of the last inequality becomes negative. Therefore, there exists $\delta(q,\eta) > 0$ such that the *infimum* of $p_\Lambda^{(2)}(q,\rho,\eta)$ over ρ cannot be achieved if $v\rho - (\mu+uq)_+ < \delta(q,\eta)$, i.e. $\rho < \tilde{\rho}_1(q,\eta) := ((\mu+uq)_+ + \delta(q,\eta))/v$.

It is clear that if $0 < q_0 < \infty$, then $\inf_{q\leq q_0}\delta(q,\eta) > 0$.

Suppose now that $\rho > \tilde{\rho}_1(q,\eta)$ and take $v\rho > \max(2\mu, 2q+2)$. Then for $k \in \Lambda^*$ one has $E(k,q,\rho) > \max(\epsilon(k),1)$. Therefore, using

$$0 \leq \frac{f(k,\rho)}{E(k,q,\rho)} - 1 \leq \frac{|h(k,q)|}{E(k,q,\rho)} \leq uq|\lambda(k)|,$$

we obtain the estimate

$$\begin{aligned}
\frac{\partial p_\Lambda^{(2)}}{\partial\rho}(q,\rho,\eta) &= -\frac{v}{V}\sum_{k\in\Lambda^*}\left\{\frac{1}{\exp(\beta E(k,q,\rho))-1} + \frac{1}{2}\coth\frac{1}{2}\beta E(k,q,\rho)\left(\frac{f(k,\rho)}{E(k,q,\rho)}-1\right)\right\}\\
&- \frac{v|\eta|^2}{(f(0,\rho)-uq)^2} + v\rho \qquad\qquad (4.16)\\
&\geq -\frac{v}{V}\sum_{k\in\Lambda^*}\frac{1}{\exp[\beta\max(\epsilon(k),1)]-1} - \frac{v}{2V}uq\sum_{k\in\Lambda^*}|\lambda(k)| - \frac{v|\eta|^2}{\delta(q,\eta)^2} + v\rho \ .
\end{aligned}$$

Making use of (1.4), this implies that there exists a volume V_0 independent of q and ρ, and $K(q,\eta) > 0$ such that if $V > V_0$, then

$$\frac{\partial p_\Lambda^{(2)}}{\partial\rho}(q,\rho,\eta) \geq -K(q,\eta) + v\rho \,,$$

and therefore, if ρ is large enough, then $\dfrac{\partial p_\Lambda^{(2)}}{\partial\rho}(q,\rho,\eta) > 0$. As a consequence, there is $\tilde\rho_2(q,\eta)$ such that the *infimum* of $p_\Lambda^{(2)}(q,\rho,\eta)$ is attained in the interval $(\tilde\rho_1(q,\eta),\tilde\rho_2(q,\eta))$. If $\bar\rho_\Lambda(q)$ is a value of ρ at which the *infimum* is attained, then $\dfrac{\partial p_\Lambda^{(2)}}{\partial\rho}(q,\bar\rho_\Lambda(q),\eta) = 0$.
Let $0 < q_0 < \infty$. Then one can see that $\sup_{q\le q_0} K(q,\eta) < \infty$, and therefore we get $\sup_{q\le q_0}\tilde\rho_2(q,\eta) < \infty$. □

Lemma 4.3 *Let $\eta \neq 0$. Then there is $q_0(\eta) < \infty$ such that the supremum of $p_\Lambda^{(2)}(q,\bar\rho_\Lambda(q),\eta)$ with respect to q is attained in the interval $(0,q_0(\eta))$ for all Λ and if $\bar q_\Lambda$ is a maximizer of $p_\Lambda^{(2)}(q,\bar\rho_\Lambda(q),\eta)$, then*

$$\frac{dp_\Lambda^{(2)}}{dq}(\bar q_\Lambda,\bar\rho_\Lambda(\bar q_\Lambda),\eta) = 0\,.$$

There exists $\bar c_0(\eta)$ such that for all Λ

$$f(0,\bar\rho_\Lambda(\bar q_\Lambda)) - u\bar q_\Lambda > \bar c_0(\eta)\,,$$

if $\bar q_\Lambda$ is a maximizer of $p_\Lambda^{(2)}(q,\bar\rho_\Lambda(q),\eta)$.

Proof: Recall that $v-u := \alpha > 0$. Differentiating $p_\Lambda^{(2)}(q,\rho,\eta)$ we get

$$\frac{\partial p_\Lambda^{(2)}}{\partial q}(q,\rho,\eta) = \frac{u^2 q}{V}\sum_{k\in\Lambda^*}|\lambda(k)|^2\left\{\frac{1}{\exp(\beta E(k,q,\rho))-1}\,\frac{1}{E(k,q,\rho)} + \frac{1}{2E(k,q,\rho)}\right\} + \frac{u|\eta|^2}{(f(0,\rho)-uq)^2} - uq\,. \tag{4.17}$$

By Lemma 4.2 we have

$$\frac{dp_\Lambda^{(2)}}{dq}(q,\bar\rho_\Lambda(q),\eta) = \frac{\partial p_\Lambda^{(2)}}{\partial q}(q,\bar\rho_\Lambda(q),\eta) + \frac{\partial p_\Lambda^{(2)}}{\partial\rho}(q,\bar\rho_\Lambda(q),\eta)\frac{d\bar\rho_\Lambda(q)}{dq} = \frac{\partial p_\Lambda^{(2)}}{\partial q}(q,\bar\rho_\Lambda(q),\eta)\,,$$

since $\partial p_\Lambda^{(2)}(q,\bar\rho_\Lambda(q),\eta)/\partial\rho = 0$. Therefore, we can also write

$$\frac{dp_\Lambda^{(2)}}{dq}(q,\bar\rho_\Lambda(q),\eta) = \frac{\partial p_\Lambda^{(2)}}{\partial q}(q,\bar\rho_\Lambda(q),\eta) + \frac{\partial p_\Lambda^{(2)}}{\partial\rho}(q,\bar\rho_\Lambda(q),\eta)\,. \tag{4.18}$$

Insertion of (4.15) and (4.17) into the identity (4.18) gives

$$\begin{aligned}\frac{dp_\Lambda^{(2)}}{dq}(q,\bar\rho_\Lambda(q),\eta) \;=\; -\frac{1}{V}\sum_{k\in\Lambda^*}\Bigg\{&\frac{1}{\exp\{\beta E(k,q,\bar\rho_\Lambda(q))\}-1}\,\frac{vf(k,\bar\rho_\Lambda(q)) - u^2q|\lambda(k)|^2}{E(k,q,\bar\rho_\Lambda(q))}\\ &+\frac{1}{2}\left(\frac{vf(k,\bar\rho_\Lambda(q)) - u^2q|\lambda(k)|^2}{E(k,q,\bar\rho_\Lambda(q))} - v\right)\Bigg\}\\ &-\frac{\alpha|\eta|^2}{(f(0,\bar\rho_\Lambda(q))-uq)^2} + v\bar\rho_\Lambda(q) - uq\,. \end{aligned} \tag{4.19}$$

Then, since $f(k,\rho) > uq|\lambda(k)| \geq uq|\lambda(k)|^2$, $f(k,\rho) > E(k,q,\bar{\rho}_\Lambda(q))$ and $\alpha > 0$, by (4.19) we get the estimate

$$\frac{dp_\Lambda^{(2)}}{dq}(q,\bar{\rho}_\Lambda(q),\eta) \leq \frac{1}{2V}\sum_{k\in\Lambda^*}\frac{u^2q|\lambda(k)|^2}{E(k,q,\bar{\rho}_\Lambda(q))} - \frac{\alpha|\eta|^2}{(f(0,\bar{\rho}_\Lambda(q)) - uq)^2} + v\bar{\rho}_\Lambda(q) - uq\,. \tag{4.20}$$

Now we have

$$\begin{aligned} E^2(k,q,\rho) &= (f(k,\rho) - uq|\lambda(k)|)(f(k,\rho) + uq|\lambda(k)|) \\ &= (\epsilon(k) + \{f(0,\rho) - uq\} + uq\{1 - |\lambda(k)|\}) \\ &\qquad\qquad \times(\epsilon(k) + \{f(0,\rho) - uq\} + uq\{1 + |\lambda(k)|\}) \\ &\geq (f(0,\rho) - uq)uq\,. \end{aligned} \tag{4.21}$$

Therefore, by (1.3), (1.4) and (4.20), (4.21) we obtain

$$\frac{dp_\Lambda^{(2)}}{dq}(q,\bar{\rho}_\Lambda(q),\eta) < \frac{\mathfrak{C}\,\mathfrak{m}_\Lambda\, q^{1/2}u^{1/2}}{2V(f(0,\bar{\rho}_\Lambda(q)) - uq)^{1/2}} - \frac{\alpha|\eta|^2}{(f(0,\bar{\rho}_\Lambda(q)) - uq)^2} + f(0,\bar{\rho}_\Lambda(q)) - uq + \mu. \tag{4.22}$$

Let $\sigma_\Lambda(q) := (f(0,\bar{\rho}_\Lambda(q)) - uq)\,(\max(1,q))^{1/3}$. Then the inequality (4.22) gives

$$\frac{dp_\Lambda^{(2)}}{dq}(q,\bar{\rho}_\Lambda(q),\eta) < \frac{(\max(1,q))^{2/3}}{\sigma_\Lambda^{1/2}(q)}\left\{\frac{\mathfrak{C}\,M\,u^{1/2}}{2} - \frac{\alpha|\eta|^2}{\sigma_\Lambda^{3/2}(q)} + \sigma_\Lambda^{3/2}(q)\right\} + \mu. \tag{4.23}$$

Therefore, there exists $c_0(\eta)$ such that if $q \geq 1$ and $\sigma_\Lambda(q) < c_0(\eta)$, then $\dfrac{dp_\Lambda^{(2)}}{dq}(q,\bar{\rho}_\Lambda(q),\eta) < 0$ for all Λ. Thus for all Λ the *supremum* of $p_\Lambda^{(2)}(q,\bar{\rho}_\Lambda(q),\eta)$ over q cannot be attained in the domain defined by the condition $\sigma_\Lambda(q) < c_0(\eta)$.

Now assume that $q \geq 1$ and $\sigma_\Lambda(q) \geq c_0(\eta)$. Then, using again (4.21), we obtain from (4.17) the estimate

$$\begin{aligned} \frac{\partial p_\Lambda^{(2)}}{\partial q}(q,\bar{\rho}_\Lambda(q),\eta) &\leq K\left\{\frac{1}{(f(0,\bar{\rho}_\Lambda(q)) - uq)} + \frac{q^{1/2}}{(f(0,\bar{\rho}_\Lambda(q)) - uq)^{1/2}}\right\} \\ &\qquad\qquad + \frac{u|\eta|^2}{(f(0,\bar{\rho}_\Lambda(q)) - uq)^2} - uq \\ &\leq K\left\{\frac{q^{1/3}}{c_0(\eta)} + \frac{q^{2/3}}{c_0^{1/2}(\eta)}\right\} + \frac{u|\eta|^2q^{2/3}}{c_0^2(\eta)} - uq\,. \end{aligned} \tag{4.24}$$

Since the right-hand side of (4.24) becomes negative for large q, there is $q_0(\eta) < \infty$ such that the *supremum* of $p_\Lambda^{(2)}(q,\bar{\rho}_\Lambda(q),\eta)$ with respect to q is attained in $q < q_0(\eta)$ for all Λ. Note that from (4.17) we see that if $\bar{q}_\Lambda$ is a maximizer of $p_\Lambda^{(2)}(q,\bar{\rho}_\Lambda(q),\eta)$, then $\bar{q}_\Lambda \neq 0$, and therefore combining this with the last statement we can deduce that

$$\frac{dp_\Lambda^{(2)}}{dq}(\bar{q}_\Lambda,\bar{\rho}_\Lambda(\bar{q}_\Lambda),\eta) = 0\,. \tag{4.25}$$

Putting $\bar{c}_0(\eta) = c_0(\eta)/\{\max(1,q_0(\eta))\}^{1/3}$ finishes the proof. $\square$

Lemma 4.4 *If $\eta \neq 0$, then*

$$\lim_{\Lambda}\{p_{\Lambda}^{(2)}(\bar{q}_{\Lambda}, \bar{\rho}_{\Lambda}(\bar{q}_{\Lambda}), \eta) - p_{\Lambda}^{(1)}(\bar{q}_{\Lambda} e^{\mathrm{i}(\pi+2\psi)}, \eta)\} = 0 \ . \tag{4.26}$$

Proof: By Bogoliubov's inequality (3.16) one gets

$$\begin{aligned} 0 &\leq p_{\Lambda}^{(2)}(\bar{q}_{\Lambda}, \bar{\rho}_{\Lambda}(\bar{q}_{\Lambda}), \eta) - p_{\Lambda}^{(1)}(\bar{q}_{\Lambda} e^{\mathrm{i}(\pi+2\psi)}, \eta) \\ &= \tilde{p}_{\Lambda}^{(2)}(\bar{q}_{\Lambda} e^{\mathrm{i}(\pi+2\psi)}, \bar{\rho}_{\Lambda}(\bar{q}_{\Lambda}), \eta) - p_{\Lambda}^{(1)}(\bar{q}_{\Lambda} e^{\mathrm{i}(\pi+2\psi)}, \eta) \\ &\leq \frac{1}{2V^2} v \left\langle (N_{\Lambda} - V\bar{\rho}_{\Lambda}(\bar{q}_{\Lambda}))^2 \right\rangle_{H_{\Lambda}^{(2)}(\bar{q}_{\Lambda} e^{\mathrm{i}(\pi+2\psi)}, \bar{\rho}_{\Lambda}(\bar{q}_{\Lambda}), \eta)} \ . \end{aligned} \tag{4.27}$$

Let $\delta N_{\Lambda} := N_{\Lambda} - V\bar{\rho}_{\Lambda}(\bar{q}_{\Lambda})$ and

$$\tilde{\Delta}_{\Lambda}(\eta) := \left\langle \delta N_{\Lambda}^2 \right\rangle_{H_{\Lambda}^{(2)}(\bar{q}_{\Lambda} e^{\mathrm{i}(\pi+2\psi)}, \bar{\rho}_{\Lambda}(\bar{q}_{\Lambda}), \eta)} \ . \tag{4.28}$$

Then (4.27) implies

$$0 \leq p_{\Lambda}^{(2)}(\bar{q}_{\Lambda}, \bar{\rho}_{\Lambda}(\bar{q}_{\Lambda}), \eta) - p_{\Lambda}^{(1)}(\bar{q}_{\Lambda} e^{\mathrm{i}(\pi+2\psi)}, \eta) \leq \frac{v}{2V^2}\tilde{\Delta}_{\Lambda}(\eta) \ .$$

We want to obtain an estimate for $\tilde{\Delta}_{\Lambda}(\eta)$ in terms of V. To this end we introduce

$$\tilde{D}_{\Lambda}(\eta) = (\delta N_{\Lambda},\, \delta N_{\Lambda})_{H_{\Lambda}^{(2)}(\bar{q}_{\Lambda} e^{\mathrm{i}(\pi+2\psi)}, \bar{\rho}_{\Lambda}(\bar{q}_{\Lambda}), \eta)} \tag{4.29}$$

and calculate the derivatives

$$\begin{aligned} \frac{\partial p_{\Lambda}^{(2)}}{\partial \mu}(q, \rho, \eta) &= \frac{1}{V}\sum_{k\in\Lambda^*}\left\{ \frac{1}{\exp(\beta E(k,q,\rho)) - 1}\, \frac{f(k,\rho)}{E(k,q,\rho)} + \frac{1}{2}\left(\frac{f(k,\rho)}{E(k,q,\rho)} - 1\right)\right\} \\ &\qquad + \frac{v|\eta|^2}{(f(0,\rho) - uq)^2} \, , \end{aligned} \tag{4.30}$$

$$\frac{\partial p_{\Lambda}^{(2)}}{\partial \rho}(q, \rho, \eta) = -v\left(\frac{\partial p_{\Lambda}^{0}}{\partial \mu}(q, \rho, \eta) - \rho\right) , \tag{4.31}$$

$$\begin{aligned} &\frac{\partial^2 p_{\Lambda}^{(2)}}{\partial \mu^2}(q, \rho, \eta) = \\ &\frac{1}{V}\sum_{k\in\Lambda^*}\left\{ \frac{\beta\exp(\beta E(k,q,\rho))}{(\exp(\beta E(k,q,\rho)) - 1)^2}\, \frac{f^2(k,\rho)}{E^2(k,q,\rho)} + \frac{1}{2}\, \frac{\exp(\beta E(k,q,\rho)) + 1}{\exp(\beta E(k,q,\rho)) - 1}\, \frac{u^2 q^2 |\lambda(k)|^2}{E^3(k,q,\rho)}\right\} \\ &\qquad + \frac{2|\eta|^2}{(f(0,\rho) - uq)^3} \, . \end{aligned} \tag{4.32}$$

From (4.32), using $e^x/(e^x - 1) \leq 2(1 + 1/x)$ for $x \geq 0$ and $f^2(k,\rho) = E(k,q,\rho)^2 + u^2q^2|\lambda(k)|^2$, we get the estimate

$$\begin{aligned} \frac{\partial^2 p_{\Lambda}^{(2)}}{\partial \mu^2}(q, \rho, \eta) &\leq \frac{2}{V}\sum_{k\in\Lambda^*}\frac{1}{(\exp(\beta E(k,q,\rho)) - 1)}\left(\beta + \frac{1}{E(k,q,\rho)}\right) \\ &+ \frac{1}{V}\sum_{k\in\Lambda^*}\left\{\frac{1}{(\exp(\beta E(k,q,\rho)) - 1)}\, \frac{2\beta E(k,q,\rho) + 3}{E^3(k,q,\rho)} + \frac{1}{2E^3(k,q,\rho)}\right\} u^2 q^2 |\lambda(k)|^2 \\ &\qquad + \frac{2|\eta|^2}{(f(0,\rho) - uq)^3} \, . \end{aligned} \tag{4.33}$$

The second sum in (4.33) is bounded from above by

$$\frac{K_0}{V}\sum_{k\in\Lambda^*}\left(\frac{1}{E^3(k,q,\rho)}+\frac{1}{E^4(k,q,\rho)}\right)u^2q^2|\lambda(k)| \le C\left(\frac{1}{(f(0,\rho)-uq)^3}+\frac{1}{(f(0,\rho)-uq)^4}\right)q^2\,,$$

and the first sum (using $E^2(k,q,\rho)\ge \epsilon(k)(\epsilon(k)-\mu)$) by

$$\begin{aligned}&\frac{K_{01}}{V}\left\{\sum_{\substack{k\in\Lambda^*\\ \epsilon(k)\le 1+4|\mu|/3}}\left(\frac{1}{E(k,q,\rho)}+\frac{1}{E^2(k,q,\rho)}\right)+\sum_{\substack{k\in\Lambda^*\\ \epsilon(k)>1+4|\mu|/3}}\frac{1}{(\exp(\beta\epsilon(k)/2)-1)}\right\}\\ &\le K_{02}\left(\frac{1}{(f(0,\rho)-uq)}+\frac{1}{(f(0,\rho)-uq)^2}+1\right).\end{aligned}$$

Consequently

$$\frac{\partial^2 p_\Lambda^{(2)}}{\partial\mu^2}(\bar q_\Lambda,\bar\rho_\Lambda(\bar q_\Lambda),\eta)\le C_1\left(\frac{1}{\bar c_0(\eta)}+\frac{1}{\bar c_0^2(\eta)}+\frac{q_0^2(\eta)}{\bar c_0^3(\eta)}+\frac{q_0^2(\eta)}{\bar c_0^4(\eta)}+1\right)+\frac{2|\eta|^2}{\bar c_0^3(\eta)}\,, \tag{4.34}$$

where $\bar c_0(\eta)$ and $q_0^2(\eta)$ are as in Lemma 4.3.

By Lemma 4.3 we have $\dfrac{\partial p_\Lambda^{(2)}}{\partial\rho}(\bar q_\Lambda,\bar\rho_\Lambda(\bar q_\Lambda),\eta)=0$. Then from (4.31) one gets that

$$\bar\rho_\Lambda(\bar q_\Lambda)=\frac{\partial p_\Lambda^{(2)}}{\partial\mu}(\bar q_\Lambda,\bar\rho_\Lambda(\bar q_\Lambda),\eta)=\frac{\partial\tilde p_\Lambda^{(2)}}{\partial\mu}(\bar q_\Lambda e^{\mathrm{i}(\pi+2\psi)},\bar\rho_\Lambda(\bar q_\Lambda),\eta)=\left\langle\frac{N_\Lambda}{V}\right\rangle_{H_\Lambda^{(2)}(\bar q_\Lambda,\bar\rho_\Lambda(\bar q_\Lambda),\eta)}\,,$$

and therefore by (4.29)

$$\frac{\tilde D_\Lambda(\eta)}{V}=\frac{\partial^2\tilde p_\Lambda^{(2)}}{\partial\mu^2}(\bar q_\Lambda e^{\mathrm{i}(\pi+2\psi)},\bar\rho_\Lambda(\bar q_\Lambda),\eta)=\frac{\partial^2 p_\Lambda^{(2)}}{\partial\mu^2}(\bar q_\Lambda,\bar\rho_\Lambda(\bar q_\Lambda),\eta)\,.$$

It then follows from (4.34) that

$$\lim_\Lambda\frac{\tilde D_\Lambda(\eta)}{V^2}=0\,. \tag{4.35}$$

Now Ginibre's inequality for (4.28) and (4.29), cf. Section 3, gives

$$\begin{aligned}&\tilde\Delta_\Lambda(\eta)\le\tilde D_\Lambda(\eta)+\\ &\frac12\beta^{1/2}\left\{\tilde D_\Lambda(\eta)\right\}^{1/2}\left\{\left\langle[N_\Lambda,\ [H_\Lambda^{(2)}(\bar q_\Lambda e^{\mathrm{i}(\pi+2\psi)},\bar\rho_\Lambda(\bar q_\Lambda),\eta),N_\Lambda]]\right\rangle_{H_\Lambda^{(2)}(\bar q_\Lambda e^{\mathrm{i}(\pi+2\psi)},\bar\rho_\Lambda(\bar q_\Lambda),\eta)}\right\}^{1/2}.\end{aligned} \tag{4.36}$$

Note that here

$$\left\langle[N_\Lambda,[H_\Lambda^{(2)}(q,\rho,\eta),N_\Lambda]]\right\rangle_{H_\Lambda^{(2)}(q,\rho,\eta)}=2u\left\langle q^*Q_\Lambda+qQ_\Lambda^*\right\rangle_{H_\Lambda^{(2)}(q,\rho,\eta)}+\sqrt V\left\langle\eta a_0^*+\eta^*a_0\right\rangle_{H_\Lambda^{(2)}(q,\rho,\eta)}.$$

By differentiating the pressure we find that

$$u\left\langle q^*Q_\Lambda+qQ_\Lambda^*\right\rangle_{H_\Lambda^{(2)}(q,\rho,\eta)}=2u|q|^2V+\frac{2V}{u}\left(q\frac{\partial\tilde p_\Lambda^{(2)}}{\partial q}(q,\rho,\eta)+q^*\frac{\partial\tilde p_\Lambda^{(2)}}{\partial q^*}(q,\rho,\eta)\right),$$

so that if we define $\hat{q} := |q|e^{\mathrm{i}(\pi+2\psi)}$, then we get

$$u\,\langle \hat{q}^* Q_{\Lambda} + \hat{q}Q_{\Lambda}^* \rangle_{H_{\Lambda}^{(2)}(\hat{q},\rho,\eta)} = 2u|q|^2 V + \frac{4V}{u}\left(|q|\frac{\partial p_{\Lambda}^{(2)}}{\partial |q|}(|q|,\rho,\eta)\right).$$

An explicit calculation gives

$$\begin{aligned}\langle \eta a_0^* + \eta^* a_0 \rangle_{H_{\Lambda}^{(2)}(q,\rho,\eta)} &= \sqrt{V}\left(\eta\frac{\partial \tilde{p}_{\Lambda}^{(2)}}{\partial \eta}(q,\rho,\eta) + \eta^*\frac{\partial \tilde{p}_{\Lambda}^{(2)}}{\partial \eta^*}(q,\rho,\eta)\right) \\ &= 2|\eta|^2\sqrt{V}\left\{\frac{f(0,\rho) - u|q|\cos(\theta-2\psi)}{f^2(0,\rho) - u^2|q|^2}\right\}\end{aligned}$$

and so

$$\langle \eta a_0^* + \eta^* a_0 \rangle_{H_{\Lambda}^{(2)}(\hat{q},\rho,\eta)} = 2\sqrt{V}\left\{\frac{|\eta|^2}{f(0,\rho) - u|q|}\right\}. \tag{4.37}$$

Therefore, if $\partial p_{\Lambda}^{(2)}(|q|,\rho,\eta)/\partial|q| = 0$, then

$$\Big\langle [N_{\Lambda},\ [H_{\Lambda}^{(2)}(\hat{q},\rho,\eta), N_{\Lambda}]] \Big\rangle_{H_{\Lambda}^{(2)}(\hat{q},\rho,\eta)} = 2V\left(2u|q|^2 + \frac{|\eta|^2}{(f(0,\rho) - u|q|)}\right).$$

Thus

$$\Big\langle [N_{\Lambda},\ [H_{\Lambda}^{(2)}(\bar{q}_{\Lambda}e^{\mathrm{i}(\pi+2\psi)}, \bar{\rho}_{\Lambda}(\bar{q}_{\Lambda}),\eta), N_{\Lambda}]] \Big\rangle_{H_{\Lambda}^{(2)}(\bar{q}_{\Lambda}e^{\mathrm{i}(\pi+2\psi)},\bar{\rho}_{\Lambda}(\bar{q}_{\Lambda}),\eta)} \le 2V\left(uq_0^2(\eta) + \frac{|\eta|^2}{\bar{c}_0(\eta)}\right).$$

From (4.35), (4.36) and the last estimate we then see that

$$\lim_{\Lambda}\frac{\tilde{\Delta}_{\Lambda}(\eta)}{V^2} = 0\,,$$

completing the proof. □

Now we prove that the order of the thermodynamic limit and taking the *infimum* and *supremum* in (4.26) can be reversed.

Proof of Theorem 1.4 :

We know from Lemma 4.3 that there is $q_0(\eta) < \infty$, independent of Λ, such that for large Λ, the maximizer $\bar{q}_{\Lambda} \in [0, q_0(\eta)]$. Then it follows from Lemma 4.2 that $\delta_0(\eta) := \inf_{q\in[0,\,q_0(\eta)]} v\tilde{\rho}_1(q,\eta) - (\mu + u\bar{q})_+ > 0$ and $\tilde{\rho}_{02}(\eta) := \sup_{q\in[0,\,q_0(\eta)]} \tilde{\rho}_2(q,\eta) < \infty$. Thus $\bar{\rho}_{\Lambda}(\bar{q})$ is in $[0, \tilde{\rho}_{02}(\eta)]$ and $v\bar{\rho}_{\Lambda}(\bar{q}) - (\mu + u\bar{q})_+ > \delta_0(\eta)$. Let $G_\eta \subset \mathbb{R}_+^2$ be the compact set

$$G_\eta := \{(q,\rho) \mid 0 \le q \le q_0(\eta),\ [(\mu + uq)_+ + \delta_0(\eta)]/v \le \rho \le \tilde{\rho}_{02}(\eta)\}\,.$$

Then $(\bar{q}_{\Lambda}, \bar{\rho}_{\Lambda}(\bar{q}_{\Lambda})) \in G_\eta$. Therefore, there is a sequence Λ_n such that $(\bar{q}_{\Lambda_n}, \bar{\rho}_{\Lambda_n}(\bar{q}_{\Lambda_n})$ converges to some point $(\bar{q},\bar{\rho})$ in G_η.

The derivatives of $p_{\Lambda}^{(2)}(q,\rho,\eta)$ are uniformly bounded on G_η and therefore as $\Lambda \uparrow \mathbb{R}^\nu$, $p_{\Lambda}^{(2)}(q,\rho,\eta)$ converges uniformly to $p^{(2)}(q,\rho,\eta)$ on G_η. Thus

$$\lim_{\Lambda} p_{\Lambda}(\eta) = \lim_{n\to\infty} p_{\Lambda_n}^{(2)}(\bar{q}_{\Lambda_n}, \bar{\rho}_{\Lambda_n}(\bar{q}_{\Lambda_n}),\eta) = p^{(2)}(\bar{q},\bar{\rho},\eta)\,. \tag{4.38}$$

By repeating the arguments of Lemmas 4.2 and 4.3 and by replacing (for $V \to \infty$) the sums over k by integrals, we see that if $\bar{\bar{q}}$ maximizer of $\inf_{\rho\,:\,\sigma(q,\rho)\geq 0} p^{(2)}(q,\rho,\eta)$ with respect to q, then $0 \leq \bar{\bar{q}} \leq q_0(\eta)$ and if $\bar{\bar{\rho}}(\bar{\bar{q}})$ is a minimizer of $p^{(2)}(\bar{\bar{q}},\rho,\eta)$, then $(\bar{\bar{q}},\bar{\bar{\rho}}(\bar{\bar{q}}))$ is in G_η. Thus

$$\sup_{q\geq 0}\ \inf_{\rho\,:\,\sigma(q,\rho)\geq 0} p^{(2)}(q,\rho,\eta) = \sup_{q\in[0,q_0(\eta)]}\ \inf_{\{\rho:(q,\rho)\in G_\eta\}} p^{(2)}(q,\rho,\eta)\,.$$

Since

$$p^{(2)}_{\Lambda_n}(\bar{q}_{\Lambda_n},\bar{\rho}_{\Lambda_n}(\bar{q}_{\Lambda_n}),\eta) \leq p^{(2)}_{\Lambda_n}(\bar{q}_{\Lambda_n},\rho,\eta)$$

for ρ such that $(\bar{q}_{\Lambda_n},\rho) \in G_\eta$, we get also that

$$p^{(2)}(\bar{q},\bar{\rho},\eta) \leq p^{(2)}(\bar{q},\rho,\eta)\,,$$

for ρ such that $(\bar{q},\rho) \in G_\eta$. That is

$$p^{(2)}(\bar{q},\bar{\rho},\eta) = \inf_{\{\rho:(\bar{q},\rho)\in G_\eta\}} p^{(2)}(\bar{q},\rho,\eta)\ . \tag{4.39}$$

Similarly, for all $q \geq 0$ we have

$$p^{(2)}_{\Lambda_n}(\bar{q}_{\Lambda_n},\bar{\rho}_{\Lambda_n}(\bar{q}_{\Lambda_n}),\eta) \geq p^{(2)}_{\Lambda_n}(q,\bar{\rho}_{\Lambda_n}(q),\eta)\ . \tag{4.40}$$

If $0 \leq q \leq q_0(\eta)$, then $(q,\bar{\rho}_{\Lambda_n}(q)) \in G_\eta$ and therefore $\bar{\rho}_{\Lambda_n}(q)$ has a convergent subsequence $\bar{\rho}_{\Lambda_{n_r}}(q)$ converging to some $\hat{\rho}$, where $(q,\hat{\rho}) \in G_\eta$. Taking the limit in (4.40) we obtain

$$p^{(2)}(\bar{q},\bar{\rho},\eta) \geq p^{(2)}(q,\hat{\rho},\eta) \geq \inf_{\{\rho:(q,\rho)\in G_\eta\}} p^{(2)}(q,\rho,\eta).$$

Thus, by (4.39)

$$p^{(2)}(\bar{q},\bar{\rho},\eta) = \inf_{\{\rho:(\bar{q},\rho)\in G_\eta\}} p^{(2)}(\bar{q},\rho,\eta) \geq \inf_{\{\rho:(q,\rho)\in G_\eta\}} p^{(2)}(q,\rho,\eta)$$

for all $q \in [0,q_0(\eta)]$. Therefore

$$p^{(2)}(\bar{q},\bar{\rho},\eta) = \sup_{q\in[0,q_0(\eta)]}\ \inf_{\{\rho:(q,\rho)\in G_\eta\}} p^{(2)}(q,\rho,\eta) = \sup_{q\geq 0}\ \inf_{\rho\,:\,\sigma(q,\rho)\geq 0} p^{(2)}(q,\rho,\eta). \tag{4.41}$$

Combining the relation (4.41) with (4.38) we prove the theorem and obtain an explicit formula for the limiting value of the pressure. □

Remark 4.2 *By the definition of $\bar{\rho}_\Lambda(q)$, $\bar{q}_\Lambda$ (Lemma 4.2, 4.3) and by (4.38) we also get that for $\eta \neq 0$*

$$\begin{aligned}\lim_{n\to\infty} p^{(2)}_{\Lambda_n}(\bar{q}_{\Lambda_n},\bar{\rho}_{\Lambda_n}(\bar{q}_{\Lambda_n}),\eta) &= \lim_{n\to\infty}\sup_{q\geq 0}\inf_{\rho\geq 0} p^{(2)}_{\Lambda_n}(q,\rho,\eta) = \sup_{q\geq 0}\inf_{\rho\geq 0} p^{(2)}(q,\rho,\eta)\\ &= p^{(2)}(\bar{q},\bar{\rho},\eta)\,,\end{aligned} \tag{4.42}$$

where (cf. (4.14))

$$\begin{aligned} p^{(2)}(q,\rho,\eta) \ &= \ -\int_{\mathbb{R}^\nu}\frac{d^\nu k}{(2\pi)^\nu}\left\{\frac{1}{\beta}\ln\left[1-\exp(-\beta E(k,q,\rho))\right] + \frac{1}{2}\left(E(k,q,\rho) - f(k,\rho)\right)\right\}\\ &+ \ \frac{|\eta|^2}{f(0,\rho)-uq} - \frac{1}{2}uq^2 + \frac{1}{2}v\rho^2\,, \end{aligned} \tag{4.43}$$

and $\bar{q},\bar{\rho}$ satisfy the equations

$$\frac{\partial p^{(2)}}{\partial\rho}(q,\rho,\eta) = 0 \quad , \quad \frac{\partial p^{(2)}}{\partial q}(q,\rho,\eta) = 0\,. \tag{4.44}$$

We now show that the zero-mode η-source term can be *switched off*.

Lemma 4.5 *Thermodynamic limit of the pressure is equal to*

$$p := \lim_{\Lambda\uparrow\mathbb{R}^\nu} p_\Lambda = \lim_{\eta\to 0}\lim_{\Lambda\uparrow\mathbb{R}^\nu} p_\Lambda(\eta) \ .$$

Proof: By Bogoliubov's convexity inequality (3.16) one gets

$$-\frac{|\eta|}{\sqrt{V}}|\,\langle a_0 + a_0^*)\rangle_{H_\Lambda}\,| \le p_\Lambda - p_\Lambda(\eta) \le \frac{|\eta|}{\sqrt{V}}|\,\langle a_0 + a_0^*)\rangle_{H_\Lambda(\eta)}\,| \,,$$

that implies

$$0 \le |p_\Lambda - p_\Lambda(\eta)| \le \frac{2|\eta|}{\sqrt{V}}|\,\langle a_0^*\rangle_{H_\Lambda(\eta)}\,| \le \frac{2|\eta|}{\sqrt{V}}\,\langle a_0^* a_0\rangle^{\frac{1}{2}}_{H_\Lambda(\eta)} \le \frac{2|\eta|}{\sqrt{V}}\,\langle N_\Lambda\rangle^{\frac{1}{2}}_{H_\Lambda(\eta)} \,. \tag{4.45}$$

From Lemma B.1 and (3.2) we see that for $|\eta| \le 1$,

$$\left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda(\eta)} \le K_1 \ ,$$

where K_1 is independent of η. Thus the right-hand side of (4.45) tends to zero as η tends to zero. □

Finally we prove that the order of the limit $\eta \to 0$ and taking the *infimum* and *supremum* in (4.41) can be *reversed*.

Lemma 4.6

$$\lim_{\eta\to 0}\sup_{q\ge 0}\inf_{\rho\,:\,\sigma(q,\rho)\ge 0} p^{(2)}(q,\rho,\eta) = \sup_{q\ge 0}\inf_{\rho\,:\,\sigma(q,\rho)\ge 0} p^{(2)}(q,\rho) \ , \tag{4.46}$$

where $p^{(2)}(q,\rho) := p^{(2)}(q,\rho,0)$ is defined in (1.14).

Proof: Let $\bar\rho_\eta(q)$ be such that

$$\inf_{\rho\,:\,\sigma(q,\rho)\ge 0} p^{(2)}(q,\rho,\eta) = p^{(2)}(q,\bar\rho_\eta(q),\eta) \ ,$$

and $\bar q_\eta$ be such that

$$\sup_{q\ge 0} p^{(2)}(q,\bar\rho_\eta(q),\eta) = p^{(2)}(\bar q_\eta,\bar\rho_\eta(\bar q_\eta),\eta) \ .$$

Let

$$G_0 := \{(q,\rho) \mid q \ge 0\,, \sigma(q,\rho) \ge 0\} \ .$$

By arguments similar to the above (see proof of Theorem 1.4) we can show that these exist and that $(\bar q_\eta, \bar\rho_\eta(\bar q_\eta)) \in G_0$. We shall need the following derivative of (4.43):

$$\frac{\partial p^{(2)}}{\partial\rho}(q,\rho,\eta) = -v\int_{\mathbb{R}^\nu}\frac{d^\nu k}{(2\pi)^\nu}\left\{\frac{1}{\exp(\beta E(k,q,\rho)) - 1}\,\frac{f(k,\rho)}{E(k,q,\rho)} + \frac{1}{2}\left(\frac{f(k,\rho)}{E(k,q,\rho)} - 1\right)\right\}$$
$$-\frac{v|\eta|^2}{(f(0,\rho) - uq)^2} + v\rho \,. \tag{4.47}$$

Moreover, in the same way as in (4.17), (4.19) we also obtain:

$$\frac{dp^{(2)}}{dq}(q,\bar{\rho}_\eta(q),\eta) = \\ u^2 q \int_{\mathbb{R}^\nu} \frac{d^\nu k}{(2\pi)^\nu} |\lambda(k)|^2 \left\{ \frac{1}{\exp(\beta E(k,q,\bar{\rho}_\eta(q))) - 1} \frac{1}{E(k,q,\bar{\rho}_\eta(q))} + \frac{1}{2E(k,q,\bar{\rho}_\eta(q))} \right\} \\ + \frac{u|\eta|^2}{(f(0,\bar{\rho}_\eta(q)) - uq)^2} - uq\,, \tag{4.48}$$

and for any number t

$$\begin{aligned} \frac{dp^{(2)}}{dq}(q,\bar{\rho}_\eta(q),\eta) \;\; = \;\; & - \int_{\mathbb{R}^\nu} \frac{d^\nu k}{(2\pi)^\nu} \left\{ \frac{1}{\exp(\beta E(k,q,\bar{\rho}_\eta(q))) - 1} \frac{t\,v f(k,\bar{\rho}_\eta(q)) - u^2 q|\lambda(k)|^2}{E(k,q,\bar{\rho}_\eta(q))} \right. \\ & \left. + \frac{1}{2}\left(\frac{t\,v f(k,\bar{\rho}_\eta(q)) - u^2 q|\lambda(k)|^2}{E(k,q,\bar{\rho}_\eta(q))} - t\,v \right) \right\} \\ & - \frac{\alpha|\eta|^2}{(f(0,\bar{\rho}_\eta(q)) - uq)^2} + t\,v\bar{\rho}_\eta(q) - uq\,. \end{aligned} \tag{4.49}$$

As in (4.24), from (4.48) we get the estimate

$$\begin{aligned} \frac{dp^{(2)}}{dq}(q,\bar{\rho}_\eta(q),\eta) \;\; \leq \;\; & K \left\{ \frac{1}{(f(0,\bar{\rho}_\eta(q)) - uq)} + \frac{q^{1/2}}{(f(0,\bar{\rho}_\eta(q)) - uq)^{1/2}} \right\} \\ + \;\; & \frac{u|\eta|^2}{(f(0,\bar{\rho}_\eta(q)) - uq)^2} - uq\,. \end{aligned} \tag{4.50}$$

Therefore, if $f(0,\bar{\rho}_\eta(\bar{q}_\eta)) - u\bar{q}_\eta \geq 1$, then by the definition of $\bar{q}_\eta$ and by (4.50) we obtain

$$0 = \frac{dp^{(2)}}{dq}(\bar{q}_\eta,\bar{\rho}_\eta(\bar{q}_\eta),\eta) \leq \frac{K_1(1+\bar{q}_\eta^{1/2})}{(f(0,\bar{\rho}_\eta(\bar{q}_\eta)) - u\bar{q}_\eta)^{1/2}} - u\bar{q}_\eta\,.$$

Since the right-hand side of the last inequality must be non-negative, then

$$f(0,\bar{\rho}_\eta(\bar{q}_\eta)) - u\bar{q}_\eta \leq \frac{K_1^2(1+\bar{q}_\eta^{\frac{1}{2}})^2}{u^2\bar{q}_\eta^2}\,.$$

Similarly, if $f(0,\bar{\rho}_\eta(\bar{q}_\eta)) - u\bar{q}_\eta \leq 1$, then

$$\frac{dp^{(2)}}{dq}(\bar{q}_\eta,\bar{\rho}_\eta(\bar{q}_\eta),\eta) \leq \frac{K_2(1+\bar{q}_\eta^{1/2})}{(f(0,\bar{\rho}_\eta(\bar{q}_\eta)) - u\bar{q}_\eta)^2} - u\bar{q}_\eta\,.$$

The right-hand side of the last inequality must be positive and thus

$$f(0,\bar{\rho}_\eta(\bar{q}_\eta)) - u\bar{q}_\eta \leq \frac{K_2^{1/2}(1+\bar{q}_\eta^{1/2})^{1/2}}{u^{1/2}\bar{q}_\eta^{1/2}}\,.$$

Therefore, either

$$1 \leq f(0,\bar{\rho}_\eta(\bar{q}_\eta)) - u\bar{q}_\eta \leq \frac{K_1^2(1+\bar{q}_\eta^{1/2})^2}{u^2\bar{q}_\eta^2} \quad \text{or} \quad 0 \leq f(0,\bar{\rho}_\eta(\bar{q}_\eta)) - u\bar{q}_\eta \leq \min\left(1, \frac{K_2^{1/2}(1+\bar{q}_\eta^{1/2})^{1/2}}{u^{1/2}\bar{q}_\eta^{1/2}}\right)\,.$$

Thus the only way that $(\bar{q}_\eta, \bar{\rho}_\eta(\bar{q}_\eta))$ can escape to infinity as $\eta \to 0$ is, if either $\bar{\rho}_\eta(\bar{q}_\eta) \to \infty$ and $\bar{q}_\eta \to 0$, or if $\bar{\rho}_\eta(\bar{q}_\eta) \to \infty$, $\bar{q}_\eta \to \infty$ and $f(0,\bar{\rho}_\eta(\bar{q}_\eta)) - u\bar{q}_\eta \to 0$. Now, if $\rho \to \infty$ and $q \to 0$, the right-hand side of (4.47) tends to $+\infty$. Therefore the case $\bar{\rho}_\eta(\bar{q}_\eta) \to \infty$ and $\bar{q}_\eta \to 0$, is not possible.

Suppose now that $\bar{\rho}_\eta(\bar{q}_\eta) \to \infty$, $\bar{q}_\eta \to \infty$ and $f(0,\bar{\rho}_\eta(\bar{q}_\eta)) - u\bar{q}_\eta \to 0$. From (4.49) with $t = u/v$ we get

$$0 = \frac{dp^{(2)}}{dq}(\bar{q}_\eta, \bar{\rho}_\eta(\bar{q}_\eta), \eta) < \frac{\|\lambda\|u}{2} + u\bar{\rho}_\eta(\bar{q}_\eta)) - u\bar{q}_\eta = \frac{\|\lambda\|u}{2} + \frac{u}{v}\left(f(0,\bar{\rho}_\eta(\bar{q}_\eta)) - u\bar{q}_\eta + \mu - \alpha\bar{q}_\eta\right).$$

This contradicts our supposition and therefore $\bar{\rho}_\eta(\bar{q}_\eta)$ and $\bar{q}_\eta$ must remain *finite.*

As in (4.22) and (4.23), from (4.49) with $t = 1$, we get

$$\begin{aligned}0 = \frac{dp^{(2)}}{dq}(\bar{q}_\eta, \bar{\rho}_\eta(\bar{q}_\eta), \eta) \quad &< \quad \frac{1}{(f(0,\bar{\rho}_\eta(\bar{q}_\eta)) - u\bar{q}_\eta)^{1/2}}\left(\frac{\|\lambda\|u^{1/2}\bar{q}_\eta^{1/2}}{2} - \frac{\alpha|\eta|^2}{(f(0,\bar{\rho}_\eta(\bar{q}_\eta)) - u\bar{q}_\eta)^{3/2}}\right)\\ &\qquad + f(0,\bar{\rho}_\eta(\bar{q}_\eta)) - u\bar{q}_\eta + \mu.\end{aligned}$$

Therefore, since the right-hand side must be positive, the term

$$\frac{|\eta|^2}{(f(0,\bar{\rho}_\eta(\bar{q}_\eta)) - u\bar{q}_\eta)^{3/2}}$$

must remain bounded when $f(0,\bar{\rho}_\eta(\bar{q}_\eta)) - u\bar{q}_\eta \to 0$.

Summarizing we see that $(\bar{q}_\eta, \bar{\rho}_\eta(\bar{q}_\eta))$ must remain in a *bounded subset* of G_0 and

$$\lim_{\eta \to 0} \frac{|\eta|^2}{(f(0,\bar{\rho}_\eta(\bar{q}_\eta) - u\bar{q}_\eta)} = 0\,. \tag{4.51}$$

Since $(\bar{q}_\eta, \bar{\rho}_\eta(\bar{q}_\eta))$ remains in a bounded subset of G_0, there exists a sequence $\eta_n \to 0$ such that $(\bar{q}_{\eta_n}, \bar{\rho}_{\eta_n}(\bar{q}_{\eta_n}))$ converges to $(\bar{q}, \bar{\rho}) \in \bar{G}_0$, where $\bar{G}_0$ is the closure of G_0. Now $p^{(2)}(q, \rho)$ is continuous on $\bar{G}_0$. Thus by (4.51) we obtain

$$\begin{aligned}\lim_\Lambda p_\Lambda \quad &= \quad \lim_{n\to\infty} p^{(2)}(\bar{q}_{\eta_n}, \bar{\rho}_{\eta_n}(\bar{q}_{\eta_n}), \eta_n)\\ &= \quad \lim_{n\to\infty} p^{(2)}(\bar{q}_{\eta_n}, \bar{\rho}_{\eta_n}(\bar{q}_{\eta_n})) + \lim_{n\to\infty} \frac{|\eta|^2}{(f(0,\bar{\rho}_{\eta_n}(\bar{q}_{\eta_n}) - u\bar{q}_{\eta_n})}\\ &= \quad p^{(2)}(\bar{q}, \bar{\rho})\;.\end{aligned}$$

Now for ρ such that $(\bar{q}, \rho) \in G_0$, for large n we have $(\bar{q}_{\eta_n}, \rho) \in G_0$. Therefore, for large n we get

$$p^{(2)}(\bar{q}_{\eta_n}, \bar{\rho}_{\eta_n}(\bar{q}_{\eta_n}), \eta_n) \le p^{(2)}(\bar{q}_{\eta_n}, \rho, \eta_n)$$

and letting $n \to \infty$, we obtain for ρ such that $(\bar{q}, \rho) \in G_0$, the estimate

$$p^{(2)}(\bar{q}, \bar{\rho}) \le p^{(2)}(\bar{q}, \rho)\;.$$

That is

$$p^{(2)}(\bar{q}, \bar{\rho}) = \inf_{\{\rho : (\bar{q}, \rho) \in G_0\}} p^{(2)}(\bar{q}, \rho).$$

Similarly, for all $q \geq 0$ we have

$$p^{(2)}_{\eta_n}(\bar{q}_{\eta_n}, \bar{\rho}_{\eta_n}(\bar{q}_{\eta_n}), \eta_n) \geq p^{(2)}_{\eta_n}(q, \bar{\rho}_{\eta_n}(q), \eta_n) \ .$$

From (4.47), we see that for each $q \geq 0$, both $\bar{\rho}_\eta(q)$ and $|\eta|^2/(f(0,\bar{\rho}_\eta(q)-uq)^2$ remain bounded as $\eta \to 0$. Let $\left\{\bar{\rho}_{\eta_{n_r}}(q)\right\}_{n_r \geq 1}$ be a convergent subsequence of $\{\bar{\rho}_{\eta_n}(q)\}_{n\geq 1}$ converging to $\hat{\rho}$ say, where $(q, \hat{\rho}) \in \bar{G}_0$. By letting $r \to \infty$ we then have

$$p^{(2)}(\bar{q}, \bar{\rho}) \geq p^{(2)}(q, \hat{\rho}) \geq \inf_{\{\rho:(q,\rho)\in G_0\}} p^{(2)}(q, \rho) \ .$$

Therefore

$$p^{(2)}(\bar{q}, \bar{\rho}) = \inf_{\{\rho:(\bar{q},\rho)\in G_0\}} p^{(2)}(\bar{q}, \rho) \geq \inf_{\{\rho:(q,\rho)\in G_0\}} p^{(2)}(q, \rho),$$

for all $q \geq 0$, and thus we get the relation

$$p^{(2)}(\bar{q}, \bar{\rho}) = \sup_{q\geq 0} \inf_{\{\rho:(q,\rho)\in G_0\}} p^{(2)}(q, \rho) = \sup_{q\geq 0} \inf_{\rho\,:\,\sigma(q,\rho)\geq 0} p^{(2)}(q, \rho)$$

proving the theorem. □

Combining Theorem 1.4, Lemma 4.5 and Lemma 4.6 we get the *first* part of our main result, Theorem 1.1, (1.20).
The second part we shall consider in the next section.

5 Discussion

Let us put in Hamiltonian (1.22) the source equal $\nu = 0$ and suppose that $\eta \neq 0$. Then the corresponding *Euler-Lagrange equations*, obtained by the condition that the derivatives (4.47) and (4.48) are equal to zero, take the form

$$\rho = \frac{1}{2}\int_{\mathbb{R}^\nu} \frac{d^\nu k}{(2\pi)^\nu} \left\{\frac{f(k,\rho)}{E(k,q,\rho)} \ \coth\frac{1}{2}\beta E(k,q,\rho) - 1\right\} \ + \ \frac{|\eta|^2}{(f(0,\rho) - uq)^2} \ , \tag{5.1}$$

$$q = \frac{u\,q}{2}\int_{\mathbb{R}^\nu} \frac{d^\nu k}{(2\pi)^\nu} \frac{|\lambda(k)|^2}{E(k,q,\rho)} \ \coth\frac{1}{2}\beta E(k,q,\rho) \ + \ \frac{|\eta|^2}{(f(0,\rho) - uq)^2} \ . \tag{5.2}$$

We shall now discuss some of the consequences of these equation in relation to the existence of Bose-Einstein condensation (BEC) in the model (1.22).

(a) The solution $(\bar{\rho}_\eta(\beta, \mu), \bar{q}_\eta(\beta, \mu))$ of the equations (5.1), (5.2) always exist and is a smooth function of β, μ and η, for $\eta \neq 0$. Moreover, we can identify it with the Gibbs expectations of the corresponding observables. Since the pressure $p_\Lambda(\nu = 0, \eta)$ is a *convex* function of μ and of u, then by the *Griffiths lemma*, see e.g. [4], the corresponding derivatives converges in the thermodynamic limit to derivatives of the limiting pressure (1.30). Differentiating (1.30) with respect to μ and u and comparing these derivatives with the solutions of (5.1) and (5.2), we get

$$\lim_\Lambda \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda(\nu=0,\eta)} = \bar{\rho}_\eta(\beta, \mu), \qquad \lim_\Lambda \left\langle \frac{Q^*_\Lambda Q_\Lambda}{V^2} \right\rangle_{H_\Lambda(\nu=0,\eta)} = \bar{q}^2_\eta(\beta, \mu).$$

(b) Similarly we can show that the zero-mode BEC for $\eta \neq 0$ is given by

$$\rho_0(\eta) := \lim_\Lambda \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_\Lambda(0,\eta)} = \frac{|\eta|^2}{(f(0,\bar{\rho}_\eta) - u\bar{q}_\eta)^2}\,. \tag{5.3}$$

To obtain this result let us make a *global* gauge transformation $U_\varphi = e^{i\varphi N_\Lambda}$ of the Hamiltonian $H_\Lambda(\mu, \nu = 0, \eta) = H_\Lambda(\nu = 0, \eta) - \mu N_\Lambda$, see (1.22), with $\varphi = \arg \eta$. Then :

$$\tilde{H}_\Lambda(\mu, 0, \eta) = U_\varphi H_\Lambda(\mu, 0, \eta) U_\varphi^* = \tilde{H}_\Lambda - \mu N_\Lambda - \sqrt{V}|\eta|\,(\tilde{a}_0^* + \tilde{a}_0)\ .$$

From

$$0 = \langle [\tilde{H}_\Lambda(\mu, 0, \eta), N_\Lambda] \rangle_{\tilde{H}_\Lambda(\mu,0,\eta)} = \sqrt{V}|\eta| \langle \tilde{a}_0^* - \tilde{a}_0 \rangle_{\tilde{H}_\Lambda(\mu,0,\eta)}$$

and

$$0 \leq \langle [N_\Lambda, [\tilde{H}_\Lambda(\mu, 0, \eta), N_\Lambda]] \rangle_{\tilde{H}_\Lambda(\mu,0,\eta)} = \sqrt{V}|\eta| \langle \tilde{a}_0^* + \tilde{a}_0 \rangle_{\tilde{H}_\Lambda(\mu,0,\eta)}$$

we obtain

$$\langle \tilde{a}_0^* \rangle_{\tilde{H}_\Lambda(\mu,0,\eta)} = \langle \tilde{a}_0 \rangle_{\tilde{H}_\Lambda(\mu,0,\eta)} \geq 0. \tag{5.4}$$

Let $\delta A_0 := (\tilde{a}_0^* + \tilde{a}_0) - \langle \tilde{a}_0^* + \tilde{a}_0 \rangle_{\tilde{H}_\Lambda(\mu,0,\eta)}$. Then

$$\frac{\partial^2 p_\Lambda(\eta)}{\partial |\eta|^2} = (\delta A_0^*,\ \delta A_0)_{\tilde{H}_\Lambda(\mu,0,\eta)} \geq 0, \tag{5.5}$$

where $(\cdot\ ,\ \cdot)_{\tilde{H}_\Lambda(\mu,0,\eta)}$ denotes the Bogoliubov-Duhamel inner product with respect to the Hamiltonian $\tilde{H}_\Lambda(\mu, \nu = 0, \eta)$. Hence, the convexity (5.5) and convergence of the pressure $p_\Lambda(\eta)$ (see Theorem 1.4 and Remark 4.2) imply by the Griffiths lemma the convergence of the first derivatives to the derivative of the limiting pressure :

$$\lim_\Lambda \frac{\partial p_\Lambda(\eta)}{\partial |\eta|} = \lim_\Lambda \frac{1}{\sqrt{V}} \langle \tilde{a}_0^* + \tilde{a}_0 \rangle_{\tilde{H}_\Lambda(\mu,0,\eta)} = \frac{2|\eta|}{f(0,\bar{\rho}_\eta) - u\bar{q}_\eta}\,, \tag{5.6}$$

see (1.30), (1.31) and (4.37). Therefore, by (5.4), (5.6), and returning back to original zero-mode operators, we obtain

$$\lim_\Lambda \left\langle \frac{a_0^*}{\sqrt{V}} \right\rangle_{H_\Lambda(0,\eta)} = \frac{\eta^*}{f(0,\bar{\rho}_\eta) - u\bar{q}_\eta}\quad ,\quad \lim_\Lambda \left\langle \frac{a_0}{\sqrt{V}} \right\rangle_{H_\Lambda(0,\eta)} = \frac{\eta}{f(0,\bar{\rho}_\eta) - u\bar{q}_\eta}\,. \tag{5.7}$$

So, by (5.7) we conclude that the η - source in Hamiltonian (1.22) *breaks* the zero-mode gauge invariance creating a zero-mode macroscopic occupation with the particle density estimated from below by the Cauchy-Schwarz inequality:

$$\begin{aligned}\lim_\Lambda \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_\Lambda(0,\eta)} &\geq \lim_\Lambda \left\langle \frac{a_0^*}{\sqrt{V}} \right\rangle_{H_\Lambda(0,\eta)} \left\langle \frac{a_0}{\sqrt{V}} \right\rangle_{H_\Lambda(0,\eta)} \\ &= \frac{|\eta|^2}{(f(0,\bar{\rho}_\eta) - u\bar{q}_\eta)^2}\,.\end{aligned} \tag{5.8}$$

To prove that in fact there is an *equality* in (5.8), we consider $p_\Lambda(\eta, s)$ the pressure with $\epsilon(0)$ replaced by $\epsilon(0) - s$ with s positive and again use its convexity with respect to s. Then Griffiths lemma and that fact that $f(0,\bar{\rho}_\eta) - u\bar{q}_\eta > 0$, as soon as $\eta \neq 0$, imply, see (4.14) and (4.38):

$$\lim_\Lambda \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_\Lambda(0,\eta)} = \lim_\Lambda \left(\frac{\partial p_\Lambda(\eta, s)}{\partial s} \right)_{s=+0} \leq \left(\frac{\partial p(\eta, s)}{\partial s} \right)_{s=+0} = \frac{|\eta|^2}{(f(0,\bar{\rho}_\eta) - u\bar{q}_\eta)^2}\,.$$

Here we have used the fact that the s-dependence of $p(\eta, s)$ is only through the last term in (1.31).

(c) In the limit $\eta \to 0$ equations (5.1) and (5.2) coincide with equations (3.7) and (3.8) or (3.10) and (3.11) in [13]. There the amount of the *generalized condensate* density is denoted there by $m_0(\beta, \mu)$. By inspection this coincides with the limit of $\rho_0(\eta)$ in (5.3) as $\eta \to 0$:

$$m_0(\beta, \mu) = \lim_{\eta \to 0} \rho_0(\eta) \ .$$

In [13] we found that for m_0 to be non-zero, μ must be greater than a certain *critical value* of chemical potential $\mu_c(\beta, u, v)$. For $u = 0$, this critical chemical potential coincides with the one for the Mean-Field boson gas (1.10), namely $\mu_c(\beta, u = 0, v) = v\rho_c(\beta)$, where $\rho_c(\beta)$ is the *critical density* for the Perfect Bose-gas, see e.g. [5].

(d) It was shown in [13] that the phase diagram is quite complicated. Subject to these Euler-Lagrange equations the expressions for the pressure given in [13] equation (2.11) and at the top of page 438, are the same as $p^{(2)}(q, \rho)$ in (1.14). (We warn the reader that in these equations for the pressure in [13] there is a misprint and a term is missing.) There we were able to solve the problem only for some values of u and v, see Fig. 2 in [13]. For example (5.2) shows that for $u > 0$ (*attraction* in the BCS part of the PBH (1.8)) the existence of the generalized Bose condensate $m_0 \neq 0$ causes an *abnormal boson pairing*:

$$\lim_{\eta \to 0} \lim_{\Lambda} \frac{1}{2} \langle Q_\Lambda^* + Q_\Lambda \rangle_{H_\Lambda(0,\eta)} = \lim_{\eta \to 0} \bar{q}_\eta(\beta, \mu) \neq 0 \,. \tag{5.9}$$

This is because, for $u > 0$, equation (5.2) cannot have the trivial solution $\bar{q}_\eta = 0$ when the *generalized* condensate

$$m_0(\beta, \mu) = \lim_{\eta \to 0} \frac{|\eta|^2}{(f(0,\bar{\rho}_\eta) - u\bar{q}_\eta)^2} \neq 0 \,. \tag{5.10}$$

Note that on the other hand the equations (5.1) and (5.2) allow the possibility that $m_0 = 0$ without $\lim_{\eta \to 0} \bar{q}_\eta = 0$. This "two-stage" condensation is possible only when $u > 0$ and it is similar to that discussed in [13].

(e) As in [13] we interpret the spectrum (1.15) of the effective Hamiltonian

$$\varepsilon_{\text{excit}}(k) := \lim_{\eta \to 0} E(k,\overline{q}_\eta,\overline{\rho}_\eta) \,, \tag{5.11}$$

as the *spectrum of excitations* for the PBH (1.8). Our analysis of the Euler-Lagrange equations (5.1), (5.2) (as well as (5.13), (5.14 below) shows that there *no gap* in this spectrum as soon as there is the Bose condensation (5.10):

$$\lim_{k \to 0} \varepsilon_{\text{excit}}(k) = \lim_{k \to 0} \lim_{\eta \to 0} (\epsilon(k) - \mu + \overline{\rho}_\eta - |u\overline{q}_\eta \lambda^*(k)|) = 0. \tag{5.12}$$

This conclusion is again in agreement with [13].

(f) The case of *repulsion* ($u \leq 0$) in the BCS part of the PBH (1.8) is quite different. In this case the pressure coincides with the *mean-field* one ($u = 0$) and we always have for the *boson paring*: $\lim_{\eta \to 0} \bar{q}_\eta(\beta, \mu) = 0$. The first property was derived in great generality in [13].

To make a contact with the variational principle proved in this paper, let us change notation and replace u by $-w$, with $w \geq 0$. The Euler-Lagrange equations, (5.1)and (5.2), become

$$\rho = \frac{1}{2}\int_{\mathbb{R}^\nu} \frac{d^\nu k}{(2\pi)^\nu} \left\{ \frac{f(k,\rho)}{E(k,q,\rho)} \coth \frac{1}{2}\beta E(k,q,\rho) - 1 \right\} + \frac{|\eta|^2}{(f(0,\ \rho) + wq)^2} , \quad (5.13)$$

$$q = \frac{(-w)q}{2}\int_{\mathbb{R}^\nu} \frac{d^\nu k}{(2\pi)^\nu} \frac{|\lambda(k)|^2}{E(k,q,\rho)} \coth \frac{1}{2}\beta E(k,q,\rho) + \frac{|\eta|^2}{(f(0,\ \rho) + wq)^2} . \quad (5.14)$$

Since the solutions $\bar{\rho}_\eta(\beta,\mu)$, $\bar{q}_\eta(\beta,\mu)$ of equations (5.13), (5.14) must satisfy the condition $\sigma(\bar{q}_\eta, \bar{\rho}_\eta) \geq 0$, one gets by (1.19) the estimate

$$f(0,\ \bar{\rho}_\eta) + w\bar{q}_\eta \geq 2w\bar{q}_\eta . \quad (5.15)$$

Note that the first term in the right-hand side of (5.14) is negative. Therefore, by (5.15) we obtain

$$\bar{q}_\eta(\beta,\mu) < \frac{|\eta|^2}{(f(0,\ \rho) + w\bar{q}_\eta(\beta,\mu))^2} < \frac{|\eta|^2}{(2w\bar{q}_\eta(\beta,\mu))^2} \quad \text{or} \quad \bar{q}_\eta(\beta,\mu) < \frac{|\eta|^{2/3}}{(2w)^{2/3}} .$$

This implies that in the limit $\eta \to 0$ the equation (5.14) may have only a *trivial* solution:

$$\lim_{\eta\to 0} \bar{q}_\eta(\beta,\mu) = 0 , \quad (5.16)$$

and

$$\lim_{\eta\to 0} \frac{|\eta|^2}{(f(0,\ \bar{\rho}_\eta) + w\bar{q}_\eta)^2} = 0 . \quad (5.17)$$

Let $\rho_c(\beta)$ be the critical density for the *Perfect* Bose Gas: $w = v = 0$, see (1.9) or (1.10),

$$\rho_c(\beta) := \int_{\mathbb{R}^\nu} \frac{d^\nu k}{(2\pi)^\nu} \frac{1}{e^{\beta\epsilon(k)} - 1}.$$

For $\mu \leq v\rho_c(\beta)$, limits (5.16) and (5.17) imply that as $\eta \to 0$ the solution of equation (5.13) tends to $\hat{\rho}(\beta,\mu)$ the solution of the corresponding equation for the Mean-Field model (1.10):

$$\rho = \int_{\mathbb{R}^\nu} \frac{d^\nu k}{(2\pi)^\nu} \frac{1}{e^{\beta(\epsilon(k)-\mu+v\rho)} - 1} ,$$

and the pressure

$$p^w(\beta,\mu) := \lim_{\eta\to 0} \inf_{\rho:\,\sigma(q,\rho)\geq 0} \inf_{q\geq 0} p^{(2)}(q,\rho,\eta) = \inf_{\rho:\,\sigma(0,\rho)\geq 0} p^{(2)}(0,\rho,0) = p^{(2)}(0,\hat{\rho}(\beta,\mu))$$

coincides with the mean-field pressure, see (1.14) and [5]. On the other hand, if $\rho > \mu/v$, then from (5.13) we obtain for any $\varepsilon > 0$ and η is sufficiently small

$$\frac{\mu}{v} < \bar{\rho}_\eta(\beta,\mu) = \rho_c(\beta) + \varepsilon ,$$

giving a contradiction for $\mu > v\rho_c(\beta)$. This means that in this case equations (5.13) and (5.14) are *inconsistent* and the minimum point must lie on the *boundary* of the allowed range on the ρ-q plane. This boundary consists of the two lines $q = 0$ and $\rho = (\mu + wq)/v$. Minimizing the pressure on the first line is equivalent to solving the variational problem in the mean-field case. This was done in [5] where one sees that the minimum is attained at

a point which tends to $\rho = \mu/v$ as $\eta \to 0$. On the other boundary $\rho = (\mu + wq)/v$ similar calculations show that the minimizer also tends to $(\rho = \mu/v, q = 0)$. Thus the pressure again coincides with the with the mean-field pressure.
This proves the *second* part of our main result for repulsive BCS interaction in the PBH, Theorem 1.1, (1.21).

We end with the following remark concerning BEC in the PBH model. Though the pressure of the model with the PB Hamiltonian for $w > 0$ coincides with the one for $w = 0$, it is an open question wether these models *coincide* completely. As has been shown in [32]-[34] a similar type of *diagonal* quadratic repulsion is able to change the type of Bose condensation, from condensation in the *zero mode* (type I) to *generalized* van den Berg-Lewis-Pulé condensation [30] out of the zero mode *without altering* the pressure. Therefore, the analysis of the Bose condensate structure in the PBH model requires a more detailed study of the corresponding quantum Gibbs states. This is beyond the scope of the present paper.

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Appendix A: Commutators

By (1.9) and (1.22) we have

$$\begin{aligned}
[H_\Lambda(\nu,\eta) - \mu N_\Lambda, Q_\Lambda] &= (-2)\sum_{k\in\Lambda^*}(\epsilon(k)-\mu)\lambda(k)A_k + \frac{2u}{V}\sum_{k\in\Lambda^*}|\lambda(k)|^2\left(N_k+\frac{1}{2}\right) Q_\Lambda \\
&- \frac{v}{V}(N_\Lambda Q_\Lambda \;+\; Q_\Lambda N_\Lambda) + 4\nu\sum_{k\in\Lambda^*}|\lambda(k)|^2\left(N_k+\frac{1}{2}\right) - 2\sqrt{V}\eta a_0\,,
\end{aligned}$$

and

$$\begin{aligned}
[Q_\Lambda^*, [H_\Lambda(\nu,\eta) - \mu N_\Lambda, Q_\Lambda]] &= 8\sum_{k\in\Lambda^*}(\epsilon(k)-\mu)|\lambda(k)|^2\left(N_k+\frac{1}{2}\right) \\
&- \frac{4u}{V}\left\{\sum_{k\in\Lambda^*}|\lambda(k)|^2\lambda^*(k)A_k^*\,Q_\Lambda + 2\left[\sum_{k\in\Lambda^*}|\lambda(k)|^2\left(N_k+\frac{1}{2}\right)\right]^2\right\} \\
&+ \frac{4v}{V}\left\{Q_\Lambda^*Q_\Lambda + 2\sum_{k\in\Lambda^*}|\lambda(k)|^2\left(N_k+\frac{1}{2}\right)(N_\Lambda+1)\right\} \\
&- 8\nu\sum_{k\in\Lambda^*}|\lambda(k)|^2\lambda^*(k)A_k^* + 4\sqrt{V}\eta\lambda(0)a_0^*\,. \qquad (A.1)
\end{aligned}$$

Using (1.5) and (1.6) we see that the first term in (A.1) is bounded by

$$8(\mathfrak{c}_\Lambda + |\mu|)\langle N_\Lambda\rangle + 4\mathfrak{n}_\Lambda + 4|\mu|\mathfrak{m}_\Lambda\,,$$

where $\langle\cdot\rangle := \langle\cdot\rangle_{H_\Lambda(\nu,\eta)}$. Recall that Lemma 2.1 gives

$$Q_\Lambda^*Q_\Lambda \le N_\Lambda^2 + MVN_\Lambda$$

and as in (2.3) we get $A_k A_k^* \leq N_k N_{-k} + 3(N_k + N_{-k}) + 2$. Using these we obtain

$$\begin{aligned}
\sum_{k\in\Lambda^*} |\lambda(k)|^3 |\langle A_k^*\, Q_\Lambda\rangle| &\leq \sum_{k\in\Lambda^*} |\lambda\ (k)| \langle A_k A_k^*\rangle^{1/2} \langle\, Q_\Lambda^* Q_\Lambda\rangle^{1/2} \\
&\leq \langle N_\Lambda^2 + MVN_\Lambda\rangle^{1/2} \left(\sum_{k\in\Lambda^*} |\lambda(k)|\right)^{1/2} \left(\sum_{k\in\Lambda^*} |\lambda(k)| \langle A_k A_k^*\rangle\right)^{1/2} \\
&\leq \langle N_\Lambda^2 + MVN_\Lambda\rangle^{1/2} \mathfrak{m}_\Lambda^{1/2} \left(\sum_{k\in\Lambda^*} |\lambda(k)| \langle N_k N_{-k} + 3(N_k + N_{-k}) + 2\rangle\right)^{1/2} \\
&\leq \langle N_\Lambda^2 + MVN_\Lambda\rangle^{1/2} \mathfrak{m}_\Lambda^{1/2} \left(\langle N_\Lambda^2 + 6N_\Lambda + 2\mathfrak{m}_\Lambda\rangle\right)^{1/2},
\end{aligned}$$

and independently we have

$$\sum_{k\in\Lambda^*} |\lambda(k)|^2 \left(N_k + \frac{1}{2}\right) \leq \sum_{k\in\Lambda^*} |\lambda(k)| \left(N_k + \frac{1}{2}\right) \leq N_\Lambda + \frac{\mathfrak{m}_\Lambda}{2}$$

and

$$\sum_{k\in\Lambda^*} |\lambda(k)|^2 \left(N_k + \frac{1}{2}\right)(N_\Lambda + 1) \leq \sum_{k\in\Lambda^*} |\lambda(k)| \left(N_k + \frac{1}{2}\right)(N_\Lambda + 1) \leq \left(N_\Lambda + \frac{\mathfrak{m}_\Lambda}{2}\right)(N_\Lambda + 1),$$

which gives estimates for the second and the third terms in (A.1). We now bound the penultimate term in (A.1).

$$\begin{aligned}
\sum_{k\in\Lambda^*} |\lambda(k)|^3 |\, \langle A_k^*\rangle\, | &\leq \sum_{k\in\Lambda^*} |\lambda(k)| |\, \langle A_k^*\rangle\, | \leq \sum_{k\in\Lambda^*} \langle N_{-k}\rangle^{1/2} \langle N_k + 1\rangle^{1/2} |\lambda(k)|^{1/2} \\
&\leq \left(\sum_{k\in\Lambda^*} \langle N_{-k}\rangle\right)^{1/2} \left(\sum_{k\in\Lambda^*} |\lambda(k)|\, \langle N_k + 1\rangle\right)^{1/2} \leq \langle N_\Lambda\rangle^{1/2} \left(\langle N_\Lambda\rangle + \mathfrak{m}_\Lambda\right)^{1/2}.
\end{aligned}$$

Finally for the last term we have

$$|\, \langle a_0^*\rangle\, | \leq \langle N_0\rangle^{1/2} \leq \langle N_\Lambda\rangle^{1/2}\ .$$

Putting these bounds together we get

$$\begin{aligned}
\langle [Q_\Lambda^*, [H_\Lambda(\nu,\eta) - \mu N_\Lambda, Q_\Lambda]]\rangle \leq\ & 8(\mathfrak{c}_\Lambda + |\mu|)\langle N_\Lambda\rangle + 4\mathfrak{n}_\Lambda + 4|\mu|\mathfrak{m}_\Lambda \\
& + \frac{4u}{V} \langle N_\Lambda^2 + MVN_\Lambda\rangle^{1/2} \mathfrak{m}_\Lambda^{1/2} \left(\langle N_\Lambda^2 + 6N_\Lambda + 2\mathfrak{m}_\Lambda\rangle\right)^{1/2} \\
& + \frac{8u}{V} \left(N_\Lambda + \frac{\mathfrak{m}_\Lambda}{2}\right)^2 + \frac{4v}{V}\left(N_\Lambda^2 + MVN_\Lambda\right) \\
& + \frac{8v}{V}\left(N_\Lambda + \frac{\mathfrak{m}_\Lambda}{2}\right)(N_\Lambda + 1) \\
& + 8\, \langle N_\Lambda\rangle^{1/2} \left(\langle N_\Lambda\rangle + \mathfrak{m}_\Lambda\right)^{1/2} + 32\sqrt{V}\, \langle N_\Lambda\rangle^{1/2}\ .
\end{aligned}$$

for $|\nu| \leq 1$ and $|\eta| \leq 1$. From Lemma B.1 and (3.2) we see that for $|\nu| \leq 1$ and $|\eta| \leq 1$,

$$\left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda(\nu,\eta)} \leq K_1 \quad \text{and} \quad \left\langle \frac{N_\Lambda^2}{V^2} \right\rangle_{H_\Lambda(\nu,\eta)} \leq K_2, \tag{A.2}$$

where K_1 and K_2 are independent of ν, η. Thus

$$\langle [Q_\Lambda^*, [H_\Lambda(\nu,\eta) - \mu N_\Lambda, Q_\Lambda]]\rangle_{H_\Lambda(\nu,\eta)} \leq C\, V^{3/2} \tag{A.3}$$

for some number C.

Appendix B: Bounds

Lemma B.1 *If a Hamiltonian H_Λ satisfies the condition*

$$H_\Lambda \geq T_\Lambda + \frac{1}{2V}\gamma N_\Lambda^2 - \delta N_\Lambda - \sigma V \tag{B.1}$$

with $\gamma > 0$ then there exist constants K_1 and K_2, depending only on γ, δ, σ and μ but not on Λ, such that

$$\left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda} \leq K_1 \tag{B.2}$$

and

$$\left\langle \frac{N_\Lambda^2}{V^2} \right\rangle_{H_\Lambda} \leq K_2\,. \tag{B.3}$$

Proof: Let $p_\Lambda(\mu)$ be the pressure for H_Λ, then

$$\left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda} \leq p_\Lambda(\mu+1) - p_\Lambda(\mu) \leq p_\Lambda(\mu+1) \leq K_1,$$

where K_1 is independent of Λ by (B.1). Also for $\lambda \in [0, \gamma)$ let

$$H_\Lambda(\lambda) := H_\Lambda - \frac{1}{2V}\lambda N_\Lambda^2\,,$$

and let $p_\Lambda(\mu, \lambda)$ be the corresponding pressure. Then

$$\left\langle \frac{N_\Lambda^2}{V^2} \right\rangle_{H_\Lambda} \leq \frac{2}{\gamma}\{p_\Lambda(\mu, \gamma/2) - p_\Lambda(\mu)\} \leq \frac{2}{\gamma} p_\Lambda(\mu, \gamma/2) \leq K_2\,,$$

where K_2 is independent of Λ, again by (B.1). □

Note that by Theorem 2.1 the Hamiltonians (1.8) and (1.22) verify the condition (B.1), see estimate (3.2).

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